GEOMETRY OF THE UNIT BALL IN VARIOUS HOLOMORPHIC SPACES
Lecture 3

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H^∞ and its subspaces

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for certain subspaces X of H^∞.

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- Since $H^\infty \subset L^\infty = L^\infty(\mathbb{T})$, we begin by recalling that the extreme (and exposed) points of $\mathrm{ball}\,(L^\infty)$ are precisely the unimodular functions.
- As to $\mathrm{ball}\,(H^\infty)$, we have the following classical result.

Theorem

Let $f \in H^\infty$ and $\|f\|_\infty = 1$. TFAE:
(i) *f is an extreme point of $\mathrm{ball}\,(H^\infty)$.*
(ii) $\int_{\mathbb{T}} \log(1 - |f(\zeta)|)\,|d\zeta| = -\infty$.

- A piece of notation: given a function $\varphi \geq 0$ on $\mathbb{T}$, with $\log\varphi \in L^1(\mathbb{T})$, we write $\mathcal{O}_\varphi$ for the outer function with modulus φ. That is,

$$\mathcal{O}_\varphi(z) := \exp\left\{\frac{1}{2\pi}\int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z}\log\varphi(\zeta)\,|d\zeta|\right\}, \qquad z \in \mathbb{D}.$$

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- **Proof.** (i) $\Longrightarrow$ (ii). If (ii) fails, then $\log(1 - |f|) \in L^1$ and we put $g := \mathcal{O}_{1-|f|}$.

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- **Proof.** (i) $\Longrightarrow$ (ii). If (ii) fails, then $\log(1 - |f|) \in L^1$ and we put $g := \mathcal{O}_{1-|f|}$.
- Then $g \in H^\infty$ and $|g| = 1 - |f|$ on $\mathbb{T}$, so $\|f \pm g\|_\infty \leq 1$; and since

$$f = \frac{1}{2}(f + g) + \frac{1}{2}(f - g),$$

we see that f is non-extreme.

- (ii) $\Longrightarrow$ (i). Suppose that $\|f \pm g\|_\infty \le 1$ for some $g \in H^\infty$. Assuming (ii), we want to show that $g \equiv 0$.

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- Since
$$\int_{\mathbb{T}} \log(1 - |f(\zeta)|^2)\,|d\zeta| = -\infty,$$
we deduce that $\int_{\mathbb{T}} \log|g(\zeta)|\,|d\zeta| = -\infty$ and so $g \equiv 0$. $\square$

- **Remark.** Let $\mathcal{A} := H^\infty \cap C(\mathbb{T})$ be the *disk algebra*. The extreme points of $\mathrm{ball}\,(\mathcal{A})$ are again characterized, among the unit-norm functions $f \in \mathcal{A}$, by condition (ii).

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- The exposed points of $\mathrm{ball}\,(H^\infty)$ have also been described.

Theorem (Amar & Lederer, 1971)

Suppose $f \in H^\infty$ and $\|f\|_\infty = 1$. Then f is an exposed point of $\mathrm{ball}\,(H^\infty)$ if and only if the set $\{\zeta \in \mathbb{T} : |f(\zeta)| = 1\}$ has positive measure.

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- The proof makes use of maximal ideals, etc.
- The exposed points of $\mathrm{ball}\,(\mathcal{A})$ are characterized by the same condition (Phelps, 1965).

Functions with large spectra

- Given a set $\Lambda \subset \mathbb{Z}_+$ and a function $f \in H^\infty$, we write $f \in H^\infty(\Lambda)$ to mean that $\widehat{f}(k) = 0$ for all $k \notin \Lambda$.

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$$H^\infty(\Lambda) := \{f \in H^\infty : \operatorname{spec} f \subset \Lambda\}.$$

- The situation is especially simple if $\mathbb{Z}_+ \setminus \Lambda$ is a finite set.

Theorem

Let Λ be a subset of $\mathbb{Z}_+$ with $\#(\mathbb{Z}_+ \setminus \Lambda) < \infty$. Suppose also that $f \in H^\infty(\Lambda)$ and $\|f\|_\infty = 1$. TFAE:

(i) *f is an extreme point of $\mathrm{ball}(H^\infty(\Lambda))$.*

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- (i) $\Longrightarrow$ (ii). Assuming that (ii) fails, put $g := \mathcal{O}_{1-|f|}$. Thus, $g \in H^\infty$ and $|g| = 1 - |f|$ on $\mathbb{T}$.

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- (i) $\Longrightarrow$ (ii). Assuming that (ii) fails, put $g := \mathcal{O}_{1-|f|}$. Thus, $g \in H^\infty$ and $|g| = 1 - |f|$ on $\mathbb{T}$.
- Now let

$$m := \#(\mathbb{Z}_+ \setminus \Lambda)$$

and write $\mathcal{P}_m$ for the set of polynomials of degree at most m.

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- Consider the linear operator $T : \mathcal{P}_m \to \mathbb{C}^m$ defined by
$$Tp := \left(\widehat{(gp)}(k_1), \dots, \widehat{(gp)}(k_m)\right), \qquad p \in \mathcal{P}_m.$$

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$$Tp := \left(\widehat{(gp)}(k_1), \dots, \widehat{(gp)}(k_m)\right), \qquad p \in \mathcal{P}_m.$$
- Because $\dim \mathcal{P}_m = m + 1$, while the rank of T does not exceed m, the *rank-nullity theorem* tells us that $\mathrm{Ker}\, T$ has dimension at least 1 and is therefore nontrivial.

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- Also,

$$|f \pm gp_0| \le |f| + |g||p_0| \le |f| + |g| = 1,$$

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- The identity

$$f = \frac{1}{2}(f + gp_0) + \frac{1}{2}(f - gp_0)$$

  now shows that f is non-extreme in $\mathrm{ball}(H^\infty(\Lambda))$. $\square$

- A similar result holds for $\mathcal{A}(\Lambda) := H^\infty(\Lambda) \cap \mathcal{A}$, where $\mathcal{A}$ is the disk algebra:

Proposition

Given a set $\Lambda \subset \mathbb{Z}_+$ with $\#(\mathbb{Z}_+ \setminus \Lambda) < \infty$, the extreme points of $\mathrm{ball}(\mathcal{A}(\Lambda))$ are precisely the unit-norm functions $f \in \mathcal{A}(\Lambda)$ satisfying (ii).

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- **Question.** For which sets $\Lambda \subset \mathbb{Z}_+$ is it true that the extreme points f of $\mathrm{ball}(H^\infty(\Lambda))$ are still characterized by

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- Such Λ's should be suitably thick in $\mathbb{Z}_+$, but $\mathbb{Z}_+ \setminus \Lambda$ need not be finite.

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  that arises.
- As obvious examples of extreme points of $\mathrm{ball}(\mathcal{P}^\infty(\Lambda))$ we mention the *monomials* $z \mapsto cz^k$, with $k \in \Lambda$ and c a unimodular constant.

Functions with small spectra

- We may assume that

$$\Lambda = \{0, 1, \dots, N\} \setminus \{k_1, \dots, k_M\}$$

for some positive integers N and k_j ($j = 1, \dots, M$) with

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for some positive integers N and k_j ($j = 1, \dots, M$) with

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- When $M = 0$, the corresponding $\mathcal{P}(\Lambda)$ space reduces to $\mathcal{P}_N$ (the non-lacunary case).

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- Let $\zeta_1, \dots, \zeta_n$ be an enumeration of the (finite and nonempty) set $\{\zeta \in \mathbb{T} : |p(\zeta)| = 1\}$. Viewed as zeros of the function

$$\tau(z) := 1 - |p(z)|^2, \qquad z \in \mathbb{T}$$

(or equivalently, of the polynomial $z^N \tau$), the ζ_j's have even multiplicities, which we denote by $2\mu_1, \dots, 2\mu_n$ respectively.

- The μ_j's are therefore positive integers. We then put

$$\mu := \sum_{j=1}^{n} \mu_j \quad \text{and} \quad \gamma := \mu/2.$$

Since $z^N \tau \in \mathcal{P}_{2N}$, it follows that $\mu \le N$.

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- For each $j \in \{1, \dots, n\}$, we consider the Wronski-type matrix

$$W_j := \begin{pmatrix} \overline{\zeta}_j^{\gamma} p(\zeta_j) & \overline{\zeta}_j^{\gamma+1} p(\zeta_j) & \dots & \overline{\zeta}_j^{N-\gamma} p(\zeta_j) \\ (\overline{z}^{\gamma} p)'(\zeta_j) & \left(\overline{z}^{\gamma+1} p\right)'(\zeta_j) & \dots & \left(\overline{z}^{N-\gamma} p\right)'(\zeta_j) \\ \vdots & \vdots & \vdots & \vdots \\ (\overline{z}^{\gamma} p)^{(\mu_j-1)}(\zeta_j) & \left(\overline{z}^{\gamma+1} p\right)^{(\mu_j-1)}(\zeta_j) & \dots & \left(\overline{z}^{N-\gamma} p\right)^{(\mu_j-1)}(\zeta_j) \end{pmatrix}$$

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- We also need the real matrices

$$\mathcal{U}_j := \operatorname{Re} W_j \quad \text{and} \quad \mathcal{V}_j := \operatorname{Im} W_j \qquad (j = 1, \dots n).$$

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- (For $k < 0$ and $k > \mu$, we obviously have $\widehat{r}(k) = 0$.)

Functions with small spectra

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- Namely, we introduce the $M \times (N - \mu + 1)$ matrix

$$\mathcal{R} := \begin{pmatrix} \widehat{r}(k_1) & \widehat{r}(k_1 - 1) & \dots & \widehat{r}(k_1 - N + \mu) \\ \vdots & \vdots & \vdots & \vdots \\ \widehat{r}(k_M) & \widehat{r}(k_M - 1) & \dots & \widehat{r}(k_M - N + \mu) \end{pmatrix}.$$

along with the real matrices

$$\mathcal{A} := \operatorname{Re} \mathcal{R} \quad \text{and} \quad \mathcal{B} := \operatorname{Im} \mathcal{R}.$$

Functions with small spectra

- Finally, we define the block matrix

$$\mathcal{M} = \mathcal{M}_\Lambda(p) := \begin{pmatrix} \mathcal{A} & -\mathcal{B} \\ \mathcal{B} & \mathcal{A} \\ \mathcal{U}_1 & \mathcal{V}_1 \\ \vdots & \vdots \\ \mathcal{U}_n & \mathcal{V}_n \end{pmatrix},$$

which has $2M + \mu$ rows and $2(N - \mu + 1)$ columns.

- Now, the result is:

Theorem

Given a finite set $\Lambda \subset \mathbb{Z}_+$ as above, suppose that p is a unit-norm polynomial in $\mathcal{P}^\infty(\Lambda)$ distinct from a monomial. Then p is an extreme point of $\mathrm{ball}(\mathcal{P}^\infty(\Lambda))$ if and only if $\operatorname{rank} \mathcal{M}_\Lambda(p) = 2(N - \mu + 1)$.

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- **Remark.** Even when $\Lambda = \{0, 1, 2\}$, the rank condition is unlikely to admit a considerable simplification. Indeed, we can find unit-norm polynomials p_1, p_2 in $\mathcal{P}_2$ satisfying

$$1 - |p_1(z)|^2 = 2\left(1 - |p_2(z)|^2\right), \qquad z \in \mathbb{T},$$

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and such that p_1 is non-extreme for $\mathrm{ball}(\mathcal{P}_2)$, while p_2 is extreme.

- E.g., take $p_1(z) = \frac{1}{2}(z^2 + 1)$ and $p_2(z) = \frac{1}{2\sqrt{2}}(z^2 + 2iz + 1)$.

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- (4) What happens between H^1 and L^1? More precisely, for which sets $E \subset \mathbb{Z}_-$ does the unit ball of

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- (5) What are the extreme points in the model space K^∞_θ?

Last slide

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