GEOMETRY OF THE UNIT BALL IN VARIOUS HOLOMORPHIC SPACES
Lecture 2

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Exposed points

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- (This means that x is the only point of contact between a certain hyperplane and the ball.)

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- Now if $x = \frac{1}{2}(y + z)$ for some $y, z \in \mathrm{ball}\,(X)$, then we have

$$1 = \phi(x) = \frac{1}{2}\left(\phi(y) + \phi(z)\right),$$

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- Because 1 is an extreme point of $\overline{\mathbb{D}}$, we see that $\phi(y) = \phi(z) = 1$.
- And since x is an exposed point of $\mathrm{ball}\,(X)$, with exposing functional ϕ, it follows that $y = z = x$. $\square$

- Now suppose that (Ω, μ) is a measure space and X is a subspace of $L^1 = L^1(\Omega, \mu)$, endowed with the usual norm $\|f\|_1 := \int_\Omega |f| d\mu$.

Theorem

Let $f \in X$ be a function with $\|f\|_1 = 1$ satisfying $f \neq 0$ a.e. on Ω. TFAE:
(i) *f is an exposed point of $\mathrm{ball}\,(X)$.*
(ii) *Whenever $h : \Omega \to [0, \infty)$ is a measurable function with $fh \in X$, we have $h = \mathrm{const}$ a.e. on Ω.*

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- **Remarks.** (1) If $h \in L^\infty_{\mathbb{R}}$ and $fh \in X$, then $h_0 := h + \|h\|_\infty$ satisfies $h_0 \geq 0$ and $fh_0 \in X$. Thus, condition (ii) is stronger (as it should be!) than its counterpart that arises for extreme points.

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- (2) Condition (ii) means that f is uniquely determined, among the unit-norm functions in X, by its argument $f/|f|$.

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- Now, if there is a nonconstant function $h \geq 0$ with $fh \in X$, then $g = fh/\|fh\|_1$ is a unit-norm function in X, $g \neq f$, and $(*)$ holds.
- Conversely, if g is a unit-norm function in X, $g \neq f$, making $(*)$ true, then $h = |g|/|f|$ is nonconstant and $fh = g \in X$. □

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(i) *f is an exposed point of* $\mathrm{ball}(X)$.
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- Here, N^+ is the *Smirnov class*, i.e.,

$$N^+ = \{u/v : \ u, v \in H^\infty, \ v \text{ outer}\}.$$

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- Indeed, if $fh (=: g) \in H^1$ for some function $h \geq 0$, then $h = g \cdot \frac{1}{f} \in H^{1/2}$. Since positive $H^{1/2}$ functions are constants, we have $h = \mathrm{const}$.

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- Indeed, letting $h = -\left(\frac{1-J}{1+J}\right)^2$, we get $fh = -(1 - J)^2 F \in H^1$. (Note that $h \geq 0$.)

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Polynomials and entire functions

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- Consider, for an example, the space $\mathcal{P}_N$ of all (analytic) polynomials of degree at most N, with norm $\|\cdot\|_1 = \|\cdot\|_{L^1(\mathbb{T})}$.
- This can be viewed as a special case of the Toeplitz kernel

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- Namely, for $\varphi = \overline{z}^{N+1}$, we have $K_1(\varphi) = \mathcal{P}_N$.
- The map $f \mapsto \overline{z\varphi f}(=: \widetilde{f})$ reduces then to the following.

- Together with a polynomial $p \in \mathcal{P}_N$ we consider its "reflection"

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- Now, for a polynomial $p \in \mathcal{P}_N$ with $\|p\|_1 = 1$, we know (from a theorem proved in Lecture 1) that *p is an extreme point of* $\mathrm{ball}\,(\mathcal{P}_N)$ *iff p and p^* have no common zeros in* $\mathbb{D}$.

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- The zeros of p and p^* being symmetric with respect to $\mathbb{T}$, this last condition can be stated in terms of p alone.

- In fact, we have

Theorem 1 (K.D., 2000)

Suppose $p \in \mathcal{P}_N$ *and* $\|p\|_1 = 1$.
(A) *Then* p *is an extreme point of* $\operatorname{ball}(\mathcal{P}_N)$ *if and only if the following conditions hold:*

(i) $|\widehat{p}(0)| + |\widehat{p}(N)| \neq 0$;
(ii) p *has no pair of symmetric zeros w. r. t.* $\mathbb{T}$.

(B) *Furthermore,* p *is an exposed point of* $\operatorname{ball}(\mathcal{P}_N)$ *if and only if it satisfies* (i), (ii) *and*
(iii) p *has no multiple zeros on* $\mathbb{T}$.

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Polynomials and entire functions

- What happens if we move from $\mathbb{T}$ to $\mathbb{R}$?
- With a function $f \in L^1(\mathbb{R})$ we associate its *Fourier transform*

$$\widehat{f}(\xi) := \int_{\mathbb{R}} e^{-ix\xi} f(x)\, dx, \qquad \xi \in \mathbb{R},$$

and the set

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- Finally, given $\sigma > 0$, we define the *Paley–Wiener space* PW^1_σ by

$$PW^1_\sigma := \{f \in L^1(\mathbb{R}) : \operatorname{spec} f \subset [-\sigma, \sigma]\}.$$

- The nonperiodic counterpart of Theorem 1 is:

Theorem 2 (K.D., 2000)

Let $\sigma > 0$. Suppose $f \in PW^1_\sigma$ and $\|f\|_1 = 1$.
(A) *Then f is an extreme point of $\mathrm{ball}\,(PW^1_\sigma)$ if and only if the following conditions hold:*

(i) *At least one of the points $\pm\sigma$ is in $\mathrm{spec}\, f$;*
(ii) *f has no pair of symmetric zeros w. r. t. $\mathbb{R}$.*

(B) *Furthermore, f is an exposed point of $\mathrm{ball}\,(PW^1_\sigma)$ if and only if it satisfies (i), (ii), as well as*
(iii) *f has no multiple zeros on $\mathbb{R}$;*
(iv) *$\int_{\mathbb{R}} |f(x)|h(x)dx = \infty$ whenever h is a nonconstant entire function of exponential type 0 that satisfies $h \geq 0$ on $\mathbb{R}$.*

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- Here, a first step was made by A. Ulanovskii and I. Zlotnikov who studied the case of

$$S = [-\sigma, -\rho] \cup [\rho, \sigma], \qquad 0 < \rho < \sigma.$$

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- Going back to $\mathbb{T}$: the space $\mathcal{P}(\Lambda)$ of "fewnomials" with spectrum in an (arbitrary) finite set $\Lambda \subset \mathbb{Z}_+$ has been studied, and the two types of points in $\mathrm{ball}\,(\mathcal{P}(\Lambda))$ have been characterized.

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- **Problem:** Characterize (at least) the extreme points of $\mathrm{ball}(H^1_{\mathcal{K}})$.

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- That is, I is writable (possibly after multiplication by a unimodular constant) as

$$I(z) = \prod_{j=1}^{m} \frac{z - a_j}{1 - \overline{a}_j z},$$

where $0 \le m \le M$ and $a_1, \dots, a_m \in \mathbb{D}$.

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- In addition, there is an interplay between the two factors, I and F, which we now describe.

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- Consider the (outer) function

$$F_0(z) := F(z) \prod_{j=1}^{m} (1 - \overline{a}_j z)^{-2}$$

and its coefficients

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- Note that $F_0 \in H^1$ and so $C_k = 0$ for all $k < 0$. Put

$$A(k) := \mathrm{Re}\, C_k, \qquad B(k) := \mathrm{Im}\, C_k \qquad (k \in \mathbb{Z}).$$

Punctured H^1: results

- Now define, for $j = 1, \dots, M$ and $\ell = 0, \dots, m$, the numbers

$$A^+_{j,\ell} := A(k_j+\ell-m)+A(k_j-\ell-m), \quad B^+_{j,\ell} := B(k_j+\ell-m)+B(k_j-\ell-m),$$

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Punctured H^1: results

- Now define, for $j = 1, \dots, M$ and $\ell = 0, \dots, m$, the numbers

$$A^+_{j,\ell} := A(k_j+\ell-m)+A(k_j-\ell-m), \quad B^+_{j,\ell} := B(k_j+\ell-m)+B(k_j-\ell-m),$$

$$A^-_{j,\ell} := A(k_j+\ell-m)-A(k_j-\ell-m), \quad B^-_{j,\ell} := B(k_j+\ell-m)-B(k_j-\ell-m).$$

- From these, we build the $M \times (m+1)$ matrices

$$\mathcal{A}^+ := \left\{A^+_{j,\ell}\right\}, \qquad \mathcal{B}^+ := \left\{B^+_{j,\ell}\right\}$$

and the $M \times m$ matrices (with j as above and $\ell = 1, \dots, m$)

$$\mathcal{A}^- := \left\{A^-_{j,\ell}\right\}, \qquad \mathcal{B}^- := \left\{B^-_{j,\ell}\right\}.$$

- Finally, we need the block matrix

$$\mathfrak{M} = \mathfrak{M}_{\mathcal{K}}\left(F, \{a_j\}_{j=1}^m\right) := \begin{pmatrix} \mathcal{A}^+ & \mathcal{B}^- \\ \mathcal{B}^+ & -\mathcal{A}^- \end{pmatrix},$$

which has $2M$ rows and $2m+1$ columns.

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- Our main result now says:

Theorem

Suppose that $f \in H^1_{\mathcal{K}}$ and $\|f\|_1 = 1$. Assume further that $f = IF$, where I is inner and F is outer. Then f is an extreme point of $\mathrm{ball}(H^1_{\mathcal{K}})$ if and only if the following two conditions hold:
(a) *I is a finite Blaschke product whose degree, say m, does not exceed M.*
(b) *The matrix $\mathfrak{M} = \mathfrak{M}_{\mathcal{K}}\left(F, \{a_j\}_{j=1}^m\right)$, built as above from F and the zeros $\{a_j\}_{j=1}^m$ of I, has rank $2m$.*

- **Example:** Suppose that $\mathcal{K} = \{k\}$, where $k \in \mathbb{N}$ and $k \geq 2$.
  Let $F \in H^1$ be an outer function with $\|F\|_1 = 1$ and $\widehat{F}(k-1) = 0$; then put $f(z) := zF(z)$. Applying the theorem, we get:

  f is an extreme point of $\mathrm{ball}(H^1_{\{k\}}) \iff |\widehat{F}(k-2)| \neq |\widehat{F}(k)|$.

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- What about exposed points in $H^1_{\mathcal{K}}$? Here is a simple sufficient condition (for a general finite set $\mathcal{K} \subset \mathbb{N}$):

Proposition

If f is an extreme point of $\mathrm{ball}(H^1_{\mathcal{K}})$ and if $1/f \in L^1$, then f is an exposed point of $\mathrm{ball}(H^1_{\mathcal{K}})$.

- The proof of the theorem relies on two lemmas.

Lemma 1

Let X be a subspace of H^1. Suppose that $f \in X$ is a function with $\|f\|_1 = 1$ whose canonical factorization is $f = IF$, with I inner and F outer. The following conditions are equivalent:
(i) *f is an extreme point of $\mathrm{ball}(X)$.*
(ii) *Whenever $h \in L^\infty_{\mathbb{R}}$ and $fh \in X$, we have $h = \mathrm{const}$.*
(iii) *Whenever $G \in H^\infty$ is a function satisfying $G/I \in L^\infty_{\mathbb{R}}$ and $FG \in X$, we have $G = cI$ for some $c \in \mathbb{R}$.*

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- This will be applied with $X = H^1_{\mathcal{K}}$.

- **Definition:** Given a nonnegative integer N and a polynomial p, we say that p is *N-symmetric* if $\overline{z}^N p(z) \in \mathbb{R}$ for all $z \in \mathbb{T}$.

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$$\widehat{p}(N-k) = \overline{\widehat{p}(N+k)}$$

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- The general form of such a polynomial is

$$p(z) = \sum_{k=0}^{N-1} (\alpha_{N-k} - i\beta_{N-k})\, z^k + 2\alpha_0 z^N + \sum_{k=N+1}^{2N} (\alpha_{k-N} + i\beta_{k-N})\, z^k,$$

where $\alpha_0, \dots, \alpha_N$ and $\beta_1, \dots, \beta_N$ are real parameters.

Two lemmas

- We may therefore identify p with its *coefficient vector*

$$(\alpha_0, \alpha_1, \dots, \alpha_N, \beta_1, \dots, \beta_N) \in \mathbb{R}^{2N+1}.$$

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- Now comes

Lemma 2

Given $N \in \mathbb{Z}_+$ *and points* $a_1, \ldots, a_N \in \mathbb{D}$*, let*

$$B(z) := \prod_{j=1}^{N} \frac{z - a_j}{1 - \overline{a}_j z}.$$

The general form of a function $G \in H^\infty$ *satisfying* $G/B \in L^\infty_{\mathbb{R}}$ *is then* $G(z) := p(z) \prod_{j=1}^{N} (1 - \overline{a}_j z)^{-2}$*, where* p *is an* N*-symmetric polynomial.*

Last slide

To be continued...