GEOMETRY OF THE UNIT BALL IN VARIOUS HOLOMORPHIC SPACES
Lecture 1

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Introduction

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- In other words, x is extreme for S if it is not an interior point of any line segment contained in S.
- *A prototypical example:* If P is a convex polyhedron in $\mathbb{R}^n$, then the extreme points of P are precisely its vertices.

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- (1) A point $x \in S$ is extreme for S iff, whenever we have

$$x = \frac{1}{2}(y + z)$$

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- **Immediate observations:**
- (1) A point $x \in S$ is extreme for S iff, whenever we have

$$x = \frac{1}{2}(y + z)$$

for some $y, z \in S$, it follows that $y = z$.
- (2) Also, a point $x \in S$ is extreme for S iff the only vector $v \in V$ satisfying

$$x + v \in S \quad \text{and} \quad x - v \in S$$

is $v = 0$.

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- As a consequence, we mention the following

Corollary

Let X be a Banach space. Then

$$\mathrm{ball}\,(X^*) := \{\varphi \in X^* : \|\varphi\|_{X^*} \le 1\}$$

(i.e., the unit ball of the dual space X^*) is the weak-star closure of the convex hull of its extreme points. In particular, $\mathrm{ball}\,(X^*)$ *has extreme points.*

- Yet another important result is the Choquet(–Bishop–de Leeuw) theorem.

Theorem (Choquet)

Suppose S is a compact convex subset of a normed (or, more generally, locally convex) space V, and let E be the set of extreme points of S. Then, for each $x \in S$, there exists a probability measure σ supported on E such that

$$x = \int_E y \, d\sigma(y),$$

in the sense that $\varphi(x) = \int_E \varphi(y)\, d\sigma(y)$ for all $\varphi \in V^*$.

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- The finite-dimensional case ($V = \mathbb{R}^n$) is due to Minkowski; the measures that arise here are finite sums of point masses.

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- Now suppose that (Ω, μ) is a measure space and X is a subspace of $L^1 = L^1(\Omega, \mu)$, endowed with the usual norm $\|f\|_1 := \int_\Omega |f| d\mu$.

Theorem

Let $f \in X$ be a function with $\|f\|_1 = 1$ satisfying $f \ne 0$ a.e. on Ω. TFAE:
(i) *f is an extreme point of $\mathrm{ball}\,(X)$.*
(ii) *Whenever $h \in L^\infty_{\mathbb{R}}$ and $fh \in X$, we have $h = \mathrm{const}$ a.e. on Ω.*

Here $L^\infty_{\mathbb{R}}$ is the set of real-valued functions in $L^\infty = L^\infty(\Omega, \mu)$.

- **Remark.** We may drop the assumption that $f \neq 0$ a.e. and replace condition (ii) by the following:

  (ii′) Whenever $h \in L^\infty_{\mathbb{R}}$ and $fh \in X$, we have $h = \mathrm{const}$ a.e. on the set $\{x \in \Omega : f(x) \neq 0\}$.

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- **Proof.** (i) $\Longrightarrow$ (ii). Suppose that (ii) fails, so that there exists a nonconstant function $h \in L^\infty_{\mathbb{R}}$ with $fh \in X$.
- Letting $\alpha := \int |f|h$ and $u := h - \alpha$, we see that $u \in L^\infty_{\mathbb{R}}$, $u \neq \mathrm{const}$, $fu \in X$, and $\int |f|u = 0$.

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- Letting $\alpha := \int |f|h$ and $u := h - \alpha$, we see that $u \in L^\infty_\mathbb{R}$, $u \neq \mathrm{const}$, $fu \in X$, and $\int |f|u = 0$.
- We may assume, in addition, that $\|u\|_\infty \leq 1$ (otherwise, replace u by εu, where $\varepsilon > 0$ is suitably small).
- It follows that the functions $f_+ := f(1+u)$ and $f_- := f(1-u)$ are *distinct* elements of X. (Indeed, fu is non-null, since u is non-null.)

- Moreover, f_+ and f_- are in $\mathrm{ball}\,(X)$, because $1 \pm u \geq 0$ and so

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- The identity

$$f = \frac{1}{2}\,(f_+ + f_-)$$

now shows that f is not an extreme point of $\mathrm{ball}\,(X)$.

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- But $|1+h| + |1-h| \ge 2$ a.e. Since $\nu(\Omega) = \|f\|_1 = 1$, it follows that $|1+h| + |1-h| = 2 \;\; \nu$-a.e. (and hence μ-a.e.).

Subspaces of L^1

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- Thus, $g = cf$ (with this c) and the equality $\|f + g\|_1 = 1$ yields

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- Now, since $fh (= g) \in X$, condition (ii) tells us that $h = \mathrm{const}$ a.e. Moreover, $h \equiv c$ for some $c \in [-1, 1]$.
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- **Corollary.** *If* μ *is atomless, then* $\mathrm{ball}\,(L^1(\Omega, \mu))$ *has no extreme points.*

- In what follows, we deal with subspaces of $L^1 := L^1(\mathbb{T}, m)$, where

$$\mathbb{T} := \{\zeta \in \mathbb{C} : |\zeta| = 1\}$$

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$$\mathbb{T} := \{\zeta \in \mathbb{C} : |\zeta| = 1\}$$
and m is normalized Lebesgue measure on $\mathbb{T}$.
- The norm $\|\cdot\|_1$ is thus given by $\|f\|_1 := \int_{\mathbb{T}} |f(\zeta)|\, dm(\zeta)$.
- The *Fourier coefficients* of a function $f \in L^1$ are the numbers
$$\widehat{f}(k) := \int_{\mathbb{T}} \overline{\zeta}^k f(\zeta)\, dm(\zeta), \qquad k \in \mathbb{Z},$$
and the *spectrum* of f is the set
$$\operatorname{spec} f := \{k \in \mathbb{Z} : \widehat{f}(k) \neq 0\}.$$

H^1 and its subspaces

- The *Hardy space* H^1 is defined by

$$H^1 := \{f \in L^1 : \operatorname{spec} f \subset \mathbb{Z}_+\},$$

where $\mathbb{Z}_+ := \{0, 1, 2, \dots\}$, and equipped with norm $\|\cdot\|_1$.

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- We may also view elements of H^1 as holomorphic functions on the unit disk

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  (use the Poisson integral to extend the function from $\mathbb{T}$ to $\mathbb{D}$).
- By definition, a non-null function $F \in H^1$ is *outer* if

$$\log|F(0)| = \int_{\mathbb{T}} \log|F(\zeta)|\, dm(\zeta).$$

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- The general form of a function $f \in H^1$, $f \not\equiv 0$, is given by $f = IF$, where I is inner and F is outer.
- For subspaces of H^1, we have the following criterion.

Theorem

Let X be a subspace of H^1. Suppose $f \in X$ is a function with $\|f\|_1 = 1$ whose canonical factorization is $f = IF$, with I inner and F outer. TFAE:
(i) *f is an extreme point of $\mathrm{ball}(X)$.*
(ii) *Whenever $h \in L^\infty_{\mathbb{R}} = L^\infty_{\mathbb{R}}(\mathbb{T})$ and $fh \in X$, we have $h = \mathrm{const}$ a.e. on $\mathbb{T}$.*
(iii) *Whenever $G \in H^\infty$ is a function satisfying $G/I \in L^\infty_{\mathbb{R}}$ and $FG \in X$, we have $G = cI$ for some $c \in \mathbb{R}$.*

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- (ii) $\implies$ (iii). If (iii) fails, then there is $G \in H^\infty$ (other than a constant multiple of I) such that $G/I \in L^\infty_{\mathbb{R}}$ and $FG \in X$.

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- Put $h := G/I$. Note that $h \in L^\infty_{\mathbb{R}}$, $h \neq \mathrm{const}$ and $fh = FG \in X$, so that (ii) fails.

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- (iii) $\implies$ (ii). If (ii) fails, then we can find an $h \in L^\infty_{\mathbb{R}}$, $h \neq \mathrm{const}$, with $fh (=: g) \in X$. Now put $G := g/F$. Because g and F are both in H^1, while F is outer, it follows that $G \in N^+$ (where N^+ is the Smirnov class).

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- (iii) $\implies$ (ii). If (ii) fails, then we can find an $h \in L^\infty_{\mathbb{R}}$, $h \neq \mathrm{const}$, with $fh (=: g) \in X$. Now put $G := g/F$. Because g and F are both in H^1, while F is outer, it follows that $G \in N^+$ (where N^+ is the Smirnov class).
- Furthermore,

$$\frac{G}{I} = \frac{g}{IF} = \frac{g}{f} = h,$$

whence $G = Ih \in L^\infty$. Therefore, $G \in N^+ \cap L^\infty = H^\infty$.

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- As a quick application, we can now prove the following classical result.

Theorem A (de Leeuw–Rudin, 1958)

A function $f \in H^1$ with $\|f\|_1 = 1$ is an extreme point of $\mathrm{ball}(H^1)$ if and only if it is outer.

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- **Proof.** If $f = IF$ is outer, then $I \equiv 1$ and the only functions $G \in H^\infty$ satisfying $G/I (= G) \in L^\infty_{\mathbb{R}}$ are the constants. Thus, (iii) holds with $X = H^1$.

- Also, $FG = g \in X$. This means that (iii) fails. □
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- Conversely, if $f = IF$ is non-outer, then put $G = 1 + I^2 (\in H^\infty)$. Note that $G/I = \overline{I} + I$, which is a nonconstant function in $L^\infty_{\mathbb{R}}$, while $FG \in H^1$. Thus, (iii) fails for $X = H^1$. □

- Now let $\varphi \in L^\infty(\mathbb{T})$. For $f \in H^1$, put

$$(T_\varphi f)(z) := \int_{\mathbb{T}} \frac{\varphi(\zeta)f(\zeta)}{1 - z\bar{\zeta}} dm(\zeta), \qquad z \in \mathbb{D}.$$

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- Note also that

$$K_1(\varphi) := \{f \in H^1 : \overline{z\varphi f} \in H^1\}.$$

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- Secondly, this happens if $\varphi = \overline{\theta}$, where θ is an inner function; in this case, $K_1(\varphi)$ becomes the *model subspace*

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associated with θ.
- What are the extreme points of $\mathrm{ball}(K_1(\varphi))$?

- When stating the answer (and below), we write $\widetilde{f} := \overline{z\varphi f}$.

Theorem B (K.D., last millennium)

For a function $f \in K_1(\varphi)$ with $\|f\|_1 = 1$, TFAE:
(i) *f is an extreme point of* $\mathrm{ball}(K_1(\varphi))$.
(ii) *The inner factors of f and $\widetilde{f}$ are relatively prime (i.e., they have no nonconstant common inner divisor).*

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- Then $fu \in K_1(\varphi)$, since $\widetilde{fu} = \overline{z\varphi fu} = \widetilde{f}\,\overline{u} \in H^1$.

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- **Proof.** (i) $\Longrightarrow$ (ii). Suppose f is extreme, and let u be the GCD of I_f and $I_{\widetilde{f}}$.
- Then $fu \in K_1(\varphi)$, since $\widetilde{fu} = \overline{z\varphi fu} = \widetilde{f}\,\overline{u} \in H^1$.
- Similarly, $f\overline{u} \in K_1(\varphi)$, since $f\overline{u} \in H^1$ and $\widetilde{f\overline{u}} = \widetilde{f}\,u \in H^1$.

- Thus, $h := 2\mathfrak{R}u = u + \overline{u}$ is in $L^\infty_{\mathbb{R}}$ and $fh \in K_1(\varphi)$. Since f is extreme, it follows that $h = \mathrm{const}$ and hence $u = \mathrm{const}$.

- Thus, $h := 2\mathfrak{R}u = u + \overline{u}$ is in $L^\infty_{\mathbb{R}}$ and $fh \in K_1(\varphi)$. Since f is extreme, it follows that $h = \mathrm{const}$ and hence $u = \mathrm{const}$.
- (ii) $\Longrightarrow$ (i). Suppose that $h \in L^\infty_{\mathbb{R}}$ and $fh =: g \in K_1(\varphi)$. Then

$$h = \frac{g}{f} = \frac{\overline{g}}{\overline{f}} = \frac{\widetilde{g}}{\widetilde{f}},$$

whence $\widetilde{f}g = f\widetilde{g}$.

- Thus, $h := 2\mathfrak{R}u = u + \overline{u}$ is in $L^\infty_{\mathbb{R}}$ and $fh \in K_1(\varphi)$. Since f is extreme, it follows that $h = \mathrm{const}$ and hence $u = \mathrm{const}$.
- (ii) $\Longrightarrow$ (i). Suppose that $h \in L^\infty_{\mathbb{R}}$ and $fh =: g \in K_1(\varphi)$. Then

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- Since I_f and $I_{\widetilde{f}}$ are relatively prime, it follows that I_f divides I_g, i.e., $I_g / I_f \in H^\infty$.
- Since $h = g/f$, we now deduce that $h \in N^+ \cap L^\infty = H^\infty$. Because h is real-valued, it must be constant. □

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- **Corollary 2.** If $\varphi \not\equiv 0$, then the extreme points of $\mathrm{ball}(K_1(\varphi))$ are dense on the unit sphere of $K_1(\varphi)$.

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To be continued...