# Lecture 7

# Two Cauchy kernels

1. Case $\operatorname{Re} w_1 \neq \operatorname{Re} w_2$.
   or $|a_1| \neq |a_2|$.
2. Frame set for $\operatorname{Re} w_1 = \operatorname{Re} w_2$
   and $|a_1| = |a_2|$.
3. Multiple poles
4. Frame bounds.

1) $g(x) = \dfrac{a_1}{x - i w_1} + \dfrac{a_2}{x - i w_2}$, $(\ast)$

2) $g(x) = \dfrac{a_1}{(x - i w)} + \dfrac{a_2}{(x - i w)^2}$,

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$a_1, a_2, w_1, w_2 \in \mathbb{C}$.

$g \in L^2(\mathbb{R})$, $\quad \operatorname{Re} w_k \neq 0$,

$\operatorname{Re} w \neq 0$

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Th. 1. $g$ is $(\ast)$. If $\operatorname{Re} w_1 \neq \operatorname{Re} w_2$ or $|a_1| \neq |a_2|$, then $\mathcal{G}(g, \alpha, \beta)$ is a frame for all $\alpha\beta < 1$.

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rescaling procedure $(\beta = 1)$.

$\underline{(\beta = 1)}$, $\operatorname{Re} w_1 = \operatorname{Re} w_2$.

$|a_1| = |a_2|$

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Shift $w_k$ by imaginary number

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$a_1 = 1$, $a_2 = -e^{2\pi i b}$, $b \in [0,1)$

$w_1 = t \in \mathbb{R}$, $w_2 = t + is$, $t, s \in \mathbb{R}$.

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$t$ is not important.

1) $s \notin \mathbb{Z}$ and $\frac{s}{\alpha} \notin \mathbb{Z}$. (exceptional situation)

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$\tau = \frac{1}{\alpha} - 1$ ($\tau$ close 0, when $\alpha \to 1$).

$$\xi_0 = \frac{1}{s}\left(\frac{s}{\alpha} - b + \lceil s\tau + b - \frac{s}{\alpha} \rceil\right) \geq \tau > 0.$$

integer part

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Th. 2. Let $g(x) = \frac{1}{x - it} - \frac{e^{2\pi i b}}{x - it + s}$,

Assume $s \notin \mathbb{Z}$, $\frac{s}{\alpha} \notin \mathbb{Z}$.

$n(s,\alpha)$ - smallest natural number.

$sn\tau + \frac{s}{\alpha} \in \mathbb{Z}$. (

$n(s,\alpha) = \infty$ if no $n$ exists.

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$G(g, \alpha, 1)$ is a frame iff.

$\xi_0 + n\tau < 1$ or $\xi_0 + n\tau \geq 1$

and $\xi_0 > \tau$.

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Th. 3. (i) $s \in \mathbb{Z}$, $G(g,\alpha,1)$ is not a frame.

(ii) $s \notin \mathbb{Z}$, $\frac{s}{\alpha} \in \mathbb{Z}$, $G(g,\alpha,1)$ is not a frame.

$\iff \alpha > 1/2$, $\xi_0 \leq 1$.

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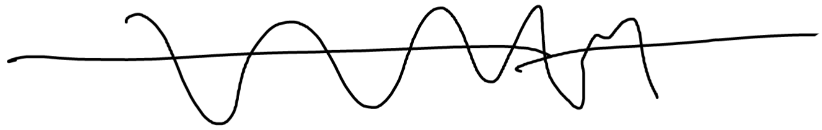
**Ex.** $s \in \mathbb{Z}$, $\quad g(x) = \frac{1}{x-it} - e^{2\pi i b} \cdot \frac{1}{(x-it)+s}$

$\hat{g}$ has periodic roots.

$\hat{g}$ 

Dual Gabor system

$$\left\{ e^{2\pi i \xi n \beta} \hat{g}(\xi - \alpha m) \right\}$$

**Th. 4.** Let $g(x) = \frac{a}{(x-i\omega)} + \frac{b}{(x-i\omega)^2}$.

$G(g, \alpha, \beta)$ is allways forms a frame for $\alpha\beta < 1$.

1) Assume $g(x) = \frac{a_1}{x - i w_1} + \frac{a_2}{x - i w_2}$,

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Main Criterion $G(g, \alpha, 1)$.

$$m_0(\xi) = a_1 e^{2\pi i \xi w_1} + a_2 e^{2\pi i \xi w_2},$$

$$m_1(\xi) = -a_1 e^{2\pi i w_1/\alpha} \cdot e^{2\pi i \xi w_1} - a_2 e^{2\pi i w_2/\alpha} \cdot e^{2\pi i \xi w_2}.$$

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MC. is a frame $\Leftrightarrow$

$$\sum_{n \in \mathbb{Z}} \int_0^{1/\alpha} \left| m_0(\xi) G(n+\xi) + m_1(\xi) G(n+\xi+1/\alpha) \right|^2 d\xi \asymp \|G\|^2, \quad \forall G \in L^2(\mathbb{R}).$$

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$$\{G(\xi_0 + \frac{n}{\alpha})\}, \quad n \in \mathbb{Z}.$$

Upper estimate obvious.

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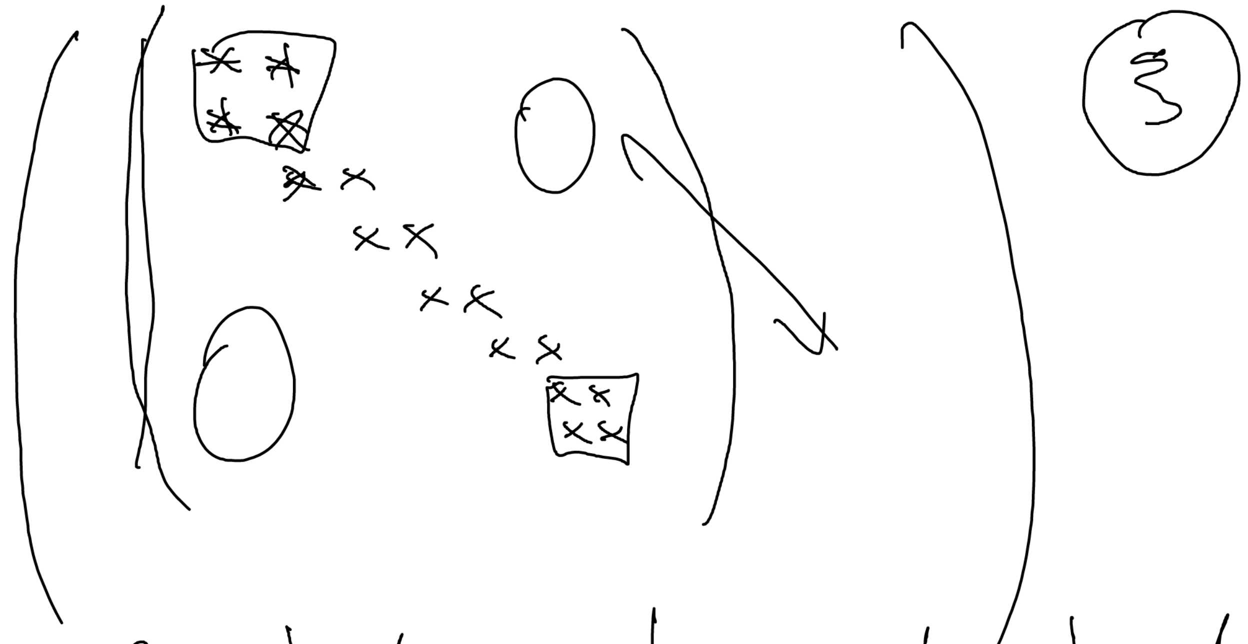

2×2 Deter. does not depend on $\xi$.

$$\det\begin{pmatrix} m_0(\xi) & m_1(\xi) \\ m_0(\xi-1) & m_1(\xi-1) \end{pmatrix} =$$

$$= a_1 a_2 e^{2\pi i \xi w_1} \cdot e^{2\pi i \xi w_2} \cdot \left(e^{2\pi i w_1} - e^{2\pi i w_2}\right)\left(e^{2\pi i w_1/\alpha} - e^{2\pi i w_2/\alpha}\right).$$

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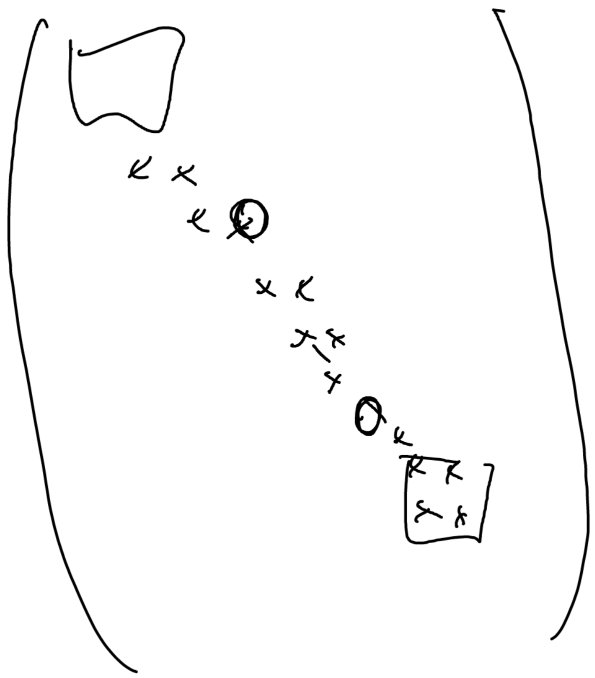
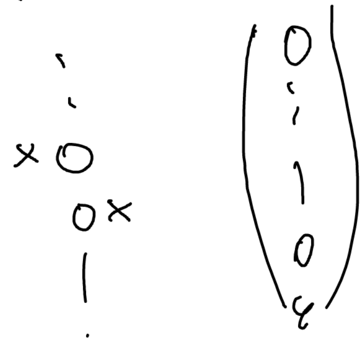
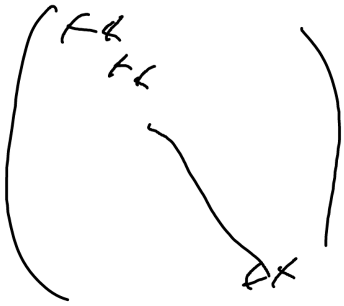
$\mathrm{Re}\, w_1 \neq \mathrm{Re}\, w_2 \Longrightarrow \det \neq 0$

rank  < maximal.



infinite two diag matrix 

## Multiple poles.

$$g(x) = \frac{a}{x - iw} + \frac{b}{(x - iw)^2}$$

$$p_0(\xi) = a - 2\pi i b\xi, \quad p_1(\xi) = e^{-2\pi w/\alpha}\left(a - 2\pi i b\xi + \frac{2\pi i b}{\alpha}\right).$$

Th. (MC. for Multiple Poles).

$G(g, \alpha, 1)$ is a frame $\Longleftrightarrow$

$$\sum_{n \in \mathbb{Z}} \int_0^{1/\alpha} \left| p_0(\xi)\, G(\xi + n) + p_1(\xi)\, G(\xi + n + 1/\alpha) \right|^2 d\xi \asymp \|G\|^2.$$

($\exists$ MC. for degree $n$.)

of same kind

$p_0(\xi), \dots \quad p_{n-1}(\xi)$.

an analogous cond. to this.

# Constants.

Th.1. $\alpha\beta=1$. in terms $m_0$ and $m_1$

$\mathcal{G}(g,1,1)$ is a frame

$\Leftrightarrow \forall \xi \in [0,1]$, $|m_0(\xi)| \neq |m_1(\xi)|$.

$m_0(\xi) = m_0(\xi, 1)$
$\alpha > 1$.

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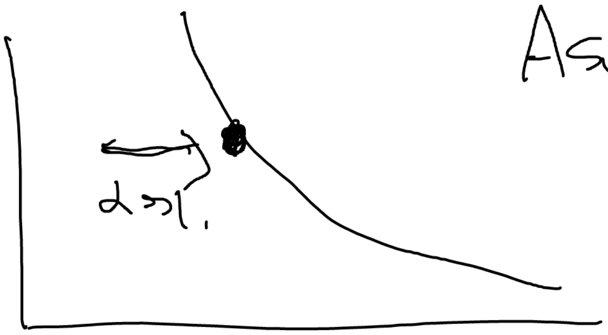



Asymptotic lower frame bound $A(\alpha)$

$\alpha \to 1$.

Th. 2. ($\mathrm{Re}\,w_1 \neq \mathrm{Re}\,w_2$) or $|a_1| \neq |a_2|$.

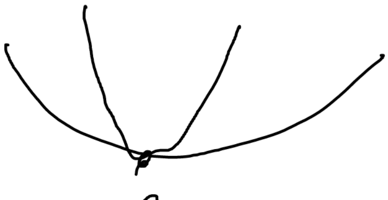
① $|m_0(\xi)| \neq |m_1(\xi)|$, then

$A(\alpha) \approx 1$.

② $\forall \xi_0$. $|m_0(\xi_0)| = |m_1(\xi_0)|$,

$>$ $<$

$\exists c_1, c_2$. $e^{-\frac{c_1}{\tau}} \leq A(\alpha) \leq e^{-\frac{c_2}{\tau}}$, $\tau = \frac{1}{\alpha} - 1$, ③.



$\xi_0$

$|m_0(\xi_0)| = |m_1(\xi_0)|$

$(|m_0(\xi)| \geq |m_1(\xi)|.$

$\tau^{c_2} \le A(\alpha) \le \tau^{c_1}$

## Ideas for new results.

1. explain a structure of non frame set (in Daubechies?).

Usually

isolated exceptional values.

2. Explain Total positive rational function.

3. Three kernells.

4. $g(x) = \dfrac{x}{(x^2+1)(x^2+4)}$, $\mathcal{F}(\alpha, \beta.\ \alpha\beta < 1,\ \alpha\beta \neq \frac{n-1}{n}, \}$?

5. Non-regular sampling (...)

6. Constants for rational functions.

7. $e^{-ax^2} g(x)$. ?

8. One Cauchy name $\dfrac{1}{x-i}$ and arbitrary TF shifts.?