Setting of the problem:

One window: $\varphi(t) = \dfrac{1}{t - iw}$?

Many lattices $\Lambda \times M$, $\Lambda, M \subset \mathbb{R}$.

$$\varphi_{\lambda,\mu}(t) = e^{-2i\pi\mu t}\,\varphi(t-\lambda);$$

$$\mathcal{G}(\Lambda, M) = \{\varphi_{\lambda,\mu}\}_{\lambda\in\Lambda,\,\mu\in M}$$

Question:

When does $\mathcal{G}(\Lambda, M)$ form a frame in $L^2(\mathbb{R})$?

Of course: This is known for $\Lambda = \alpha\mathbb{Z}$, $\mu = \beta\mathbb{Z}$.

$$\hat{\varphi}(\xi) = e^{iw\xi}\,\mathbb{1}_{(0,\infty)}(\xi) \qquad (\operatorname{Re} w > 0).$$

Janssen 1996

Gröchenig Stöckler 2010

Notation, definitions

1. Paley-Wiener space:

$\beta > 0 \Rightarrow$

$$PW_{[0,\beta]} = \left\{ f(t) = \int_0^\beta e^{2i\pi t\xi} \hat{f}(\xi)\, d\xi ,\ \hat{f} \in L^2(0,\beta) \right\}.$$

2. Sampling sequences in $PW_{[0,\beta]}$

Λ - sampling sequence $\Leftrightarrow$

$$\|f\|_{L^2} \asymp \|\{f(\lambda)\}\|_{\ell^2},\ f \in PW_{[0,\beta]}$$

Examples: • Riesz basis

• $D^-(\Lambda) > \beta$

Ortega-Seip (2008)

3. Locally finite sets:

No clusters, no gaps!

$M = \{\mu_n\}$, $\mu_n < \mu_{n+1}$

- $\sup_x \# \{M \cap (x, x+1)\} < \infty$
- $\beta_n := \mu_{n+1} - \mu_n$, $\beta(M) := \sup_n \beta_n < \infty$

Theorem: TFAE:

1) $\mathcal{G}(\Lambda, M)$ is a frame

2) M is locally finite and Λ is sampling in $PW_{[0, \beta(M)]}$.

$\operatorname{Re} w > 0$

Proof: $M = \{\mu_n\}$

$$c_{\lambda, n} = \langle \varphi_{\lambda, \mu_n}, \bar{f} \rangle =$$

$$= \frac{1}{2i\pi} \int_{-\infty}^{\infty} f(t) \frac{e^{-2i\pi\mu_n t}}{t - \lambda - iw} dt$$

$$f_k(t) = \int_{\mu_k}^{\mu_{k+1}} \hat{f}(\xi) e^{2i\pi\xi t} d\xi$$

$$f = \sum f_k, \quad \|f\|^2 = \sum \|f_k\|^2.$$

$$f_k(t) = h_k(t) e^{2i\pi\mu_k t}$$

$$h_k \in PW_{[0, \beta_k]}.$$

$$c^k_{\lambda, n} = \langle \varphi_{\lambda, \mu_n}, \bar{f_k} \rangle =$$

$$= \frac{1}{2i\pi} \int_{-\infty}^{\infty} f_k(t) \frac{e^{-2i\pi\mu_n t}}{t - \lambda - iw} dt$$

$e^{+2i\pi\mu_k t} h_k(t)$

$$= \frac{1}{2i\pi} \int_{-\infty}^{\infty} h_k(t) \frac{e^{2i\pi(\mu_k - \mu_n)t}}{t - \lambda - iw} dt.$$

$= 0 \qquad n \geq k \qquad\qquad \mu_n - \mu_k.$

$n \leq k.$

Exponent:

$$2i\pi(\mu_n - \mu_k)(\lambda + i\omega) = 2i\pi\mu_n\lambda - 2i\pi\mu_k\lambda + 2\pi(\mu_n - \mu_k)\omega$$

$$c_{\lambda,n} = e^{-2i\pi\lambda\mu_n} \sum_{k \geq n} h_k(\lambda + i\omega)\, e^{2i\pi\mu_k\lambda}\, e^{2\pi(\mu_n - \mu_k)\omega}$$

$$\underbrace{e^{-2i\pi\lambda\mu_n} c_{\lambda,n}}_{d_{\lambda n}} = e^{-2i\pi\lambda\mu_n} \sum_{k \geq n} \underbrace{\left(h_k(\lambda + i\omega)\, e^{2i\pi\mu_k\lambda}\, e^{2\pi(\mu_n - \mu_k)\omega} \right)}_{\omega_{\lambda,k}}$$

$$\vec{c}_\lambda = \{ c_{\lambda,n} \}_{n \in \mathbb{Z}}$$

$$\vec{\omega}_\lambda = \{ \omega_{\lambda,k} \}_k.$$

$$\vec{C}_n = A\,\vec{\omega}_\lambda$$

$\boxed{A \text{ invertible.}}$

$$\|\vec{C}_\lambda\|^2 \simeq \|\vec{\omega}_\lambda\|^2,$$

$$\sum \|\vec{C}_\lambda\|^2 \simeq \sum \|\vec{\omega}_\lambda\|^2.$$

$$\boxed{A = (a_{kn}) \qquad a_{kn} = \begin{cases} 0 & n < k \\ e^{2\pi i w(\mu_n - \mu_k)} & n \geq k. \end{cases}}$$

$$A = I + T \qquad \begin{pmatrix} & & T \\ & & \\ & & \end{pmatrix}$$

$$\beta_m = \mu_{m+1} - \mu_m.$$

$$\gamma_m = 2\pi w\, \beta_m$$

$$\Big(1 \quad \gamma_1 \quad \gamma_2 + \gamma_1 \quad \cdots$$

$$B = \begin{pmatrix} 0 & e^{\gamma_1} & & \\ & \ddots & e^{\gamma_2} & \\ & & \ddots & \\ & & & 0 \end{pmatrix}$$

$$B^2 = \begin{pmatrix} 0 & 0 & e^{\gamma_1+\gamma_2} & & \\ & \ddots & \ddots & e^{\gamma_2+\gamma_3} & \\ & & \ddots & \ddots & \\ & & & \ddots & 0 \\ & & & & 0 \end{pmatrix}$$

$$\boxed{A = I + \sum_{j=1}^{\infty} B^j}$$

$$- 2\pi w \cancel{2\pi w} \underbrace{(\gamma_j + \dots + \gamma_{j+N})}_{} < 1$$

$$A = (I - B)^{-1}$$

$$\Longrightarrow \quad \beta(M) \quad A$$

$$\Lambda \quad PW_{\beta(M)}$$

$$\Lambda \quad PW_{\beta_n}$$

$$n \qquad f_n \qquad \mu_n \qquad \mu_{n+1}$$

$$\| \{ h_n(\lambda + i w) \}_\lambda \|_{\ell^2}$$

$$\approx \longleftarrow \text{By } \text{invertibility of } A.$$

$$\sum | c_{\lambda, w} |^2$$

$$\approx$$

$$\| f \|^2 .$$

$$\frac{a_1}{t - iw_1} + \frac{a_2}{t - iw_2} = \varphi(t)$$

$Re\, w_1, Re\, w_2 \neq 0$ and also $Re\, w_1 \neq Re\, w_2$.

$Re\, w_1 = Re\, w_2$ and $|a_1| \neq |a_2|$

$$\varphi(t) = \frac{t}{t^2 + 1}$$

Sweat dream

$$\varphi(t) = \frac{t}{(t^2+1)(t^2+4)}$$