Lecture 5.

Near the critical hyperbola. Large densities

1. Contractive operator near the CH.
2. Non-frame systems with large densities.
3. Counterexamples.

$$g(x) = \sum_{k=1}^{N} \frac{a_k}{x - i w_k},$$

$a_k, w_k \in \mathbb{C}$, $\operatorname{Re} w_k \neq 0$.

$g \in L^2(\mathbb{R})$.

$\alpha\beta > 1$

non-frame.

frame.

$\alpha\beta < 1$

$\alpha\beta$ close 1.

Za_k type transform function.

$$\mathcal{L}(z,\xi) = \sum_{k=1}^{N} \frac{a_k e^{2\pi i \xi \omega_k}}{1 - z e^{2\pi i \omega_k/\alpha}}$$

$$= \frac{\sum_{s=0}^{N-1} m_s(\xi) z^s}{\prod_{k=1}^{N} \left(1 - z e^{2\pi i \omega_k/\alpha}\right)}$$

m_s - from Main Criterion.

Near critical hyperbola

<u>Th.</u> Let $\operatorname{Re} \omega_k > 0$ and

$$\operatorname{Re} \mathcal{L}(e^{2\pi i t}, \xi) > 0.$$

$t \in \mathbb{R}$, $\xi \in \mathbb{R}$. Then.

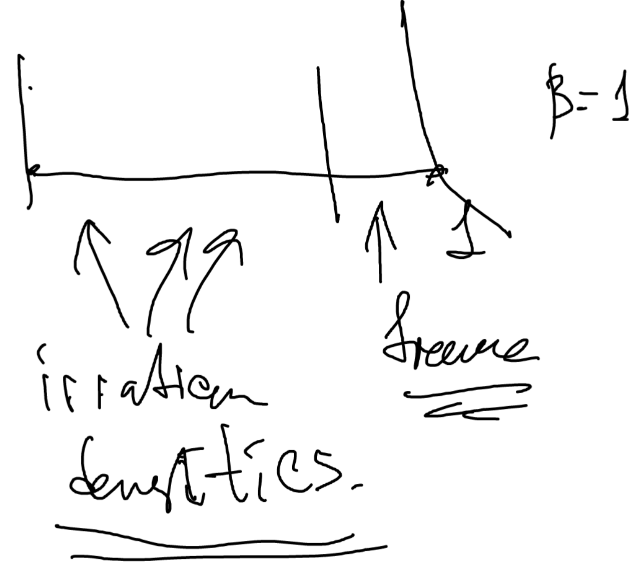
there exists $\alpha_0 < 1$ such that $G(g; \alpha, 1)$ is a frame for all $\alpha \in (\alpha_0, 1)$.

Th. Let. $\operatorname{Re} w_k < 0$. $\hat{g}(\xi) \neq 0$, $\xi > 0$.

and $\operatorname{Re} w_k \neq \operatorname{Re} w_l$ and $k \neq l$, $\operatorname{Re} Z(e^{2\pi i t}, \xi) > 0$. $t \in \mathbb{R}$. $\xi \in \mathbb{R}$. Then.

$G(g; \alpha, 1)$ is a frame for all $\alpha \in (0, 1]$ except perhaps a finite number of exceptional values.

Th. 1 t. $\hat{g}(\xi) \neq 0$.



Rem. $\forall \delta > 0$, $\exists q = q(\delta)$.

if $\alpha = \frac{p}{q}$, $q > q(\delta)$

$\alpha < 1 - \delta \Rightarrow G(g, \alpha, 1)$ is a frame.

if $\hat{g}(\xi) \neq 0$.

Corollary. Let $w_k < 0$, $a_k \in \mathbb{R}$

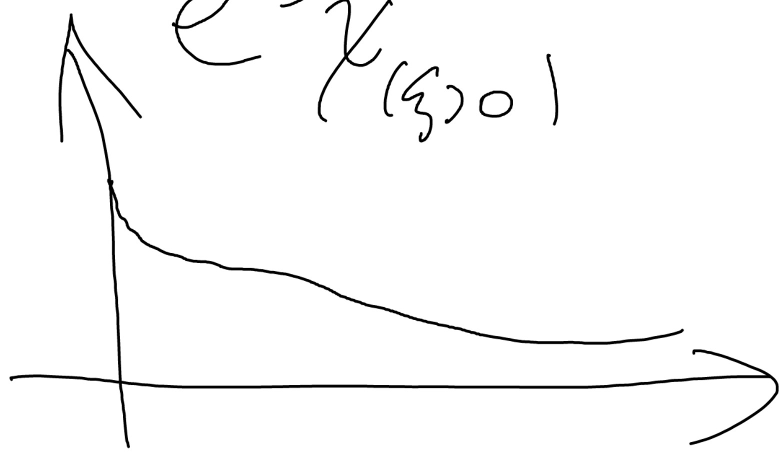
$\hat{g}$ ~ is real valued.

Assume $\hat{g}$ is positive decreasing convex function on $\mathbb{R}_+$. Then $G(g, \alpha, \beta)$ is a frame for all (α, β) suff. close to CH. $(\alpha > \alpha(\beta))$

For fixed β there is only finit numbe of exceptional α.

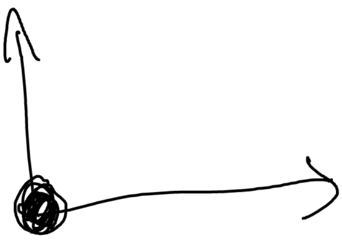
$$\frac{1}{x+i}$$

$e^{-\xi}\chi_{(\xi>0)}$



$$\hat{g}(\xi) = e^{-\xi} - \frac{1}{10}e^{-2\xi} + \frac{1}{2}e^{-3\xi}$$

Large densities.



g - fixed $\exists\, \varepsilon > 0$.

$\alpha < \varepsilon$ and $\beta < \varepsilon \Rightarrow G(g; \alpha, \beta)$ frame.

$\alpha\beta < \varepsilon$ this is not true.

g - continuous and $g(n) = 0$, $n \in \mathbb{Z}$

$G(g; 1, \beta)$ $\forall \beta > 0$.

not a frame.

$$\{e^{2\pi i n\beta x} g(x-\alpha m)\}_{n,m\in\mathbb{Z}}$$

$\underbrace{\operatorname{Re} w_k \neq \operatorname{Re} w_l,}$ $\alpha\beta$

Th. $\operatorname{Re} w_k \neq \operatorname{Re} w_\ell$, $k \neq \ell$.

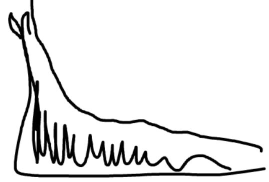
If $\alpha\beta \le \frac{1}{N}$, then $G(g;\alpha,\beta)$ is a frame.

N - number of poles

More or less sharp. $\forall N$.

rational g, $\operatorname{Re} w_k \neq \operatorname{Re} w_\ell$.

$\alpha = \frac{1}{N-1}$, $\beta = 1$, such that $G(g;\alpha,\beta)$ is not a frame.



Th. $Rew_u \neq Rew_l$. $G(q; \alpha, \beta)$ is always complete for $\alpha\beta \leq 1$.

Proofs. Similar to Herglotz case.

$$LG(\xi) = \sum_{s=0}^{N-1} G\left(\xi + \frac{s}{\alpha}\right) m_s(\xi).$$

m_s - given $\qquad \|LG\| \geq \varepsilon \|G\|$

$$LG = H.$$

$$\begin{pmatrix} * & x & x & x & * & x & x & x \\ 1 & & & & & & & \\ & 1 & & & 0 & & & \\ 0 & & 1 & & & & & \\ & & & 1 & & & & \end{pmatrix}$$

- Frobenius matrix corresponding $F_{P\{\xi\}}$.

$$p(\xi) = \frac{1}{m_{N-1}(\xi)}\left(m_0(\xi) + m_1(\xi)z + \dots + m_{N-1}(\xi)z^{N-1}\right)$$

$$\left\| F_{p\{\xi\}} F_{p\{\xi-\frac{1}{\alpha}\}} F_{p\{\xi-\frac{2}{\alpha}\}} \cdots F_{p\{\xi-\frac{\ell}{\alpha}\}} \right\| \le Cq^{\ell},$$

$q < 1$

α is close to 1. $\frac{1}{\alpha}$ is clos 1

$\{\xi - \frac{1}{\alpha}\}$

clos $\{\xi\}$.

Spectrar radius of $F_{p\{\xi\}} \le q < 1$.

$\|F^M\| \le q < 1$.

Spectral radius of F^M.

is maximal. eigenvalue.

$\boxed{m}$ = roots of $p(\xi)$

$$= \frac{1}{m_{N-1}(\xi)}\left(\sum_{s=0}^{N-1} m_s(\xi) z^s\right).$$

$$Z(z,\xi) = \frac{\sum_{s=0}^{N-1} m_s(\xi) z^s}{\prod_{k=1}^{N}\left(1 - z e^{2\pi i \omega_k/\alpha}\right)}$$

$\operatorname{Re} \omega_k > 0$, all poles $\in \mathbb{D}$.

$z = e^{-2\pi i \omega_k/\alpha} \in \mathbb{D}$.

$\operatorname{Re} Z$ on $\mathbb{T}$.

$$m_0(\xi) = \sum_{k=1}^{N} a_k e^{2\pi i \xi \omega_k}, \quad \omega_k < 0.$$

$$\overset{\shortparallel}{\hat{g}(\xi)}.$$

$$Re(z,\xi) = \sum_{k=1}^{N} Re \frac{a_k e^{2\pi i \xi w_k}}{1 - z e^{2\pi i w_k/\alpha}} =$$

$$\frac{1}{1-t} = \sum_{\ell=0}^{\infty} t^{\ell}, \quad \Big| \quad z = e^{2\pi i t}.$$

$$= \sum_{k=1}^{N} a_k e^{2\pi i \xi w_k} \sum_{\ell=0}^{\infty} Re\left(z^{\ell} e^{2\pi i \ell \frac{w_k}{\alpha}}\right)$$

$$= \sum_{n \geq 0} \underset{\hat{g}\left(\xi + \frac{n}{\alpha}\right).}{\underset{\|}{m_0\left(\xi + \frac{n}{\alpha}\right)}} \cos(nt) \geq 0$$

Large densities.

Main Criterion. $G(g;\alpha,1)$

$$\Leftrightarrow \int_0^{1/\alpha} \sum_{n\in\mathbb{Z}} \left| \sum_{s=0}^{N-1} G\left(\xi + n + \frac{s}{\alpha}\right) m_s(\xi) \right|^2 d\xi \gtrsim \|G\|^2.$$

$G \in L^2(\mathbb{R})$. $\quad \alpha \leq 1$

$G(\xi_0 + \frac{s}{\alpha})$, $s \in \mathbb{Z}$ $\quad N \geq 1$

$\xi_0 = \xi + n$, $\quad n \in \mathbb{Z}$, $\xi \in [0; 1/\alpha]$.

α is small. we have many representations

$\alpha \leq \frac{1}{N}$, there exists at least N representations

L_θ. submatrix

$$B = \begin{pmatrix} m_0(\theta) & m_1(\theta) & \dots & m_{N-1}(\theta) \\ m_0(\theta-1) & \dots & \dots & m_{N-1}(\theta-1) \\ \\ m_0(\theta-(N-1)) & \dots & \dots & m_{N-1}(\theta-(N-1)) \end{pmatrix}$$ sq matrix

$\det B \neq 0$, $\quad \theta \in [N-1, N]$.

$$\sum_{k=1}^{N} \frac{a_k e^{2\pi i \xi w_k}}{1 - z e^{2\pi i w_k/\alpha}} = \frac{\sum_{s=0}^{N-1} m_s(\xi) z^s}{\prod_{k=1}^{N} (1 - z e^{2\pi i w_k/\alpha})}.$$

$$m_s(\xi) = (-1)^s \sum_{k=1}^{N} a_k e^{2\pi i \xi w_k} \cdot A_{ks},$$

$$A_{ks} = \sum_{j_e \neq k} e^{2\pi i (w_{j_1}/\alpha + \ldots + w_{j_{k}}/e)}$$

$N=3$,

$$m_0(\xi) = a_1 e^{2\pi i \xi w_1} + a_2 e^{2\pi i \xi w_2} + a_3 e^{2\pi i \xi w_3}$$

$$-m_1(\xi) = a_1 e^{2\pi i \xi w_1} \left(e^{2\pi i w_2/\alpha} + e^{2\pi i w_3/\alpha}\right) +$$

$$+ a_2 e^{2\pi i \xi w_2} \left(e^{2\pi i w_1/\alpha} + e^{2\pi i w_3/\alpha}\right) +$$

$$+ a_3 e^{2\pi i \xi w_3} \left(e^{2\pi i w_1/\alpha} + e^{2\pi i w_2/\alpha}\right)$$

$$m_2(\xi) = a_1 e^{2\pi i \xi w_1} \cdot e^{2\pi i w_2/\alpha} \cdot e^{2\pi i w_3/\alpha}$$

$$+ a_2 e^{2\pi i \xi w_2} \cdot e^{2\pi i w_1/\alpha} \cdot e^{2\pi i w_3/\alpha}$$

$$+ a_3 e^{2\pi i \xi w_3} \cdot e^{2\pi i w_1/\alpha} \cdot e^{2\pi i w_2/\alpha}.$$

Put $u_k = e^{2\pi i w_k/\alpha}$, $A_k = e^{2\pi i \xi w_k}$,

$y_k = e^{-2\pi i w_k}$,

$$B(\xi) = \begin{pmatrix} A_1 & A_2 & \dots & A_N \\ A_1 y_1 & A_2 y_2 & \dots & A_N y_N \\ \vdots & & & \\ A_1 y_1^{N-1} & A_2 y_2^{N-1} & \dots & A_N y_N^{N-1} \end{pmatrix} \times$$

$$\times \begin{pmatrix} 1 & -\sum_{k\neq 1} u_k & \dots & (-1)^{N-1} \prod_{k\neq 1} u_k \\ 1 & \sum_{k\neq 2} u_k & & \\ \vdots & & & \\ 1 & \sum_{k\neq N} u_k & \dots & (-1)^{N-1} \prod_{k\neq N} u_k \end{pmatrix}$$

symmetric polynomials

$$= X \cdot Y$$

X - Vandermonde matrix.

$$\det Y = \pm \prod_{k\neq l} (u_k - u_l) \neq 0.$$

Counter example, $\alpha = \frac{1}{N-1}$

L_θ N, $N-1$ rows.

$$\begin{pmatrix} m_0(\theta) & \cdots & m_{N-1}(\theta) \\ \vdots & & \vdots \\ m_0(\theta-(N-2)) & \cdots & m_{N-1}(\theta-(N-2)) \end{pmatrix} \begin{pmatrix} \vdots \end{pmatrix}$$

shift by 1

Rew$_k \neq$ Rew$_\ell$, choose a_k.

Such that $\sum_{s=0}^{N-1} m_s(\theta-\ell) = 0$, $\ell = 0, \ldots, N-2$.

$$L_\theta \begin{pmatrix} \vdots \end{pmatrix} = 0, \quad M \left\{ \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 1 \\ 0 \end{pmatrix} \right., \quad \|L_\theta I_M\|$$

is small

~~is~~ is bounded

Counter examples. $Rew_k \neq Rew_l$,

for each rational $\alpha = p/q$ $\exists$

— — for irrational.

non tame system