Lecture 4

Frames for irrational Densities

1) Statement of the Thm
2) Beginning of the proof.
3) Computational part
4) App!

Th $g(x) = \sum_{k=1}^{N} \frac{a_k}{x - i w_k}$ $G(g; \alpha, 1)$-frame

iff

$$\int_0^{\frac{1}{\alpha}} \sum_{n \in \mathbb{Z}} \left| \sum_{s=0}^{N-1} G\left(n + \xi + \frac{s}{\alpha}\right) m_s(\xi) \right|^2 d\xi \asymp \|G\|^2_{L^2}$$

$$m_0(\xi) = \sum_{k=1}^{N} a_k e^{2\pi \xi w_k}$$

Th Let $g = \sum \frac{a_k}{x - i w_k}$ $\operatorname{Re} w_k \neq \operatorname{Re} w_\ell$, $k \neq \ell$.
$m_0(\xi) \neq 0$, $\xi > 0$. Then for all α, β:
$\alpha\beta < 1$ $\alpha\beta \notin \mathbb{Q}$ $G(g; \alpha, \beta)$-frame

Cor. for almost all rat func g
$G(g;\alpha,\beta)$ is a frame if $\alpha\beta<1$, $\alpha\beta\notin\mathbb{Q}$

Cor. for a.a. Balian-Low rat. f. g.
$g=\sum\frac{d_k}{x-iw_k}$ is B-L $\sum d_k=0$.
$m_0(0)=0$ $\quad$ $G(g;\alpha,\beta)$ frame if $\alpha\beta<1$, $\alpha\beta\notin\mathbb{Q}$

Assume $\operatorname{Re} w_k<0$.
One can check that $m_0(\xi)=\hat{g}(\xi)$, $\xi>0$
Cor if $g(x)=\sum\frac{d_k}{x-iw_k}$, $\operatorname{Re} w_k\neq \operatorname{Re} w_\ell$
$\operatorname{Re} w_k<0$ and $\hat{g}(\xi)\neq 0$, $\xi>0$ then
for all α,β: $\alpha\beta<1$ $\alpha\beta\notin\mathbb{Q}$ $G(g;\alpha,\beta)$ frame
Examples: $\sum\frac{d_k}{x-iw_k}$, $d_k, w_k>0$ then
for all α,β: $\alpha\beta<1$ $G(g;\alpha,\beta)$

$m_0(\xi)=\sum d_k e^{2\pi i\xi w_k}>0$

Th (Stöckler, Gröchenig)

$g(x) = \prod_{k=1}^{N} \frac{1}{1+i\delta_k x}$, $\delta_k \in \mathbb{R}\setminus\{0\}$

$\alpha\beta \geq 1 \Rightarrow G(g;\alpha,\beta)$ - frame.

Assume δ_k are distinct.

$m_0(\xi) = \sum a_k e^{2\pi\xi w_k}$ $w_k = -\frac{1}{\delta_k}$

$m_0(0) = m_0'(0) = \dots = m_0^{(N-2)}(0) = 0$

Descartes' rule of signs

$m_0(\xi) > 0$, $\xi > 0$.

Proof: $\int_0^{\frac{1}{\alpha}} \sum_{n\in\mathbb{Z}} \left| \underbrace{\sum_{s=0}^{N-1} m_s(\xi)\, G(\xi+n+\frac{s}{\alpha})}\right|^2 d\xi \asymp \|G\|^2$ $(+)$

$G(g;\alpha,1)$ $\beta = 1$ $\int_0^{\frac{1}{\alpha}} \to \int_0^1$

$\alpha \to \alpha\beta$ $\beta \to 1$ $g \to g_\beta(t) = g\left(\frac{t}{\beta}\right)$

$m_{0,\beta}(t) = \frac{m_0(t\beta)}{\beta} \neq 0$, $t > 0$

WLOG $\beta = 1$.

WLOG $\frac{1}{2} < \alpha < 1$. $G(g;2\alpha,1)$ frame $\Rightarrow$

$\Rightarrow G(g;\alpha,1)$ - frame $\underbrace{2\alpha\mathbb{Z}}\times\mathbb{Z} < \underbrace{\alpha\mathbb{Z}\times\mathbb{Z}}$

LHS of $(+)$ acts "independently" on each sequence $\{G(\xi+\frac{k}{\alpha})\}_{k\in\mathbb{Z}}$, $\xi \in [0,\frac{1}{\alpha})$

So, it is enough to prove a uniform bound for this operator restricted to (almost) all such progressions.

Fix $\zeta_0 \in (0, \frac{1}{\alpha})$

$\rho^2(\frac{1}{\alpha}\mathbb{Z} + \zeta_0)$ Put $\tau = \frac{1}{\alpha} - 1 > 0$

$$\underbrace{\sum_{s=0}^{N-1} m_s(\zeta)\, G(\zeta + n + \tfrac{s}{\alpha})}$$

$s = 0$

$m_0(\zeta)$ ~

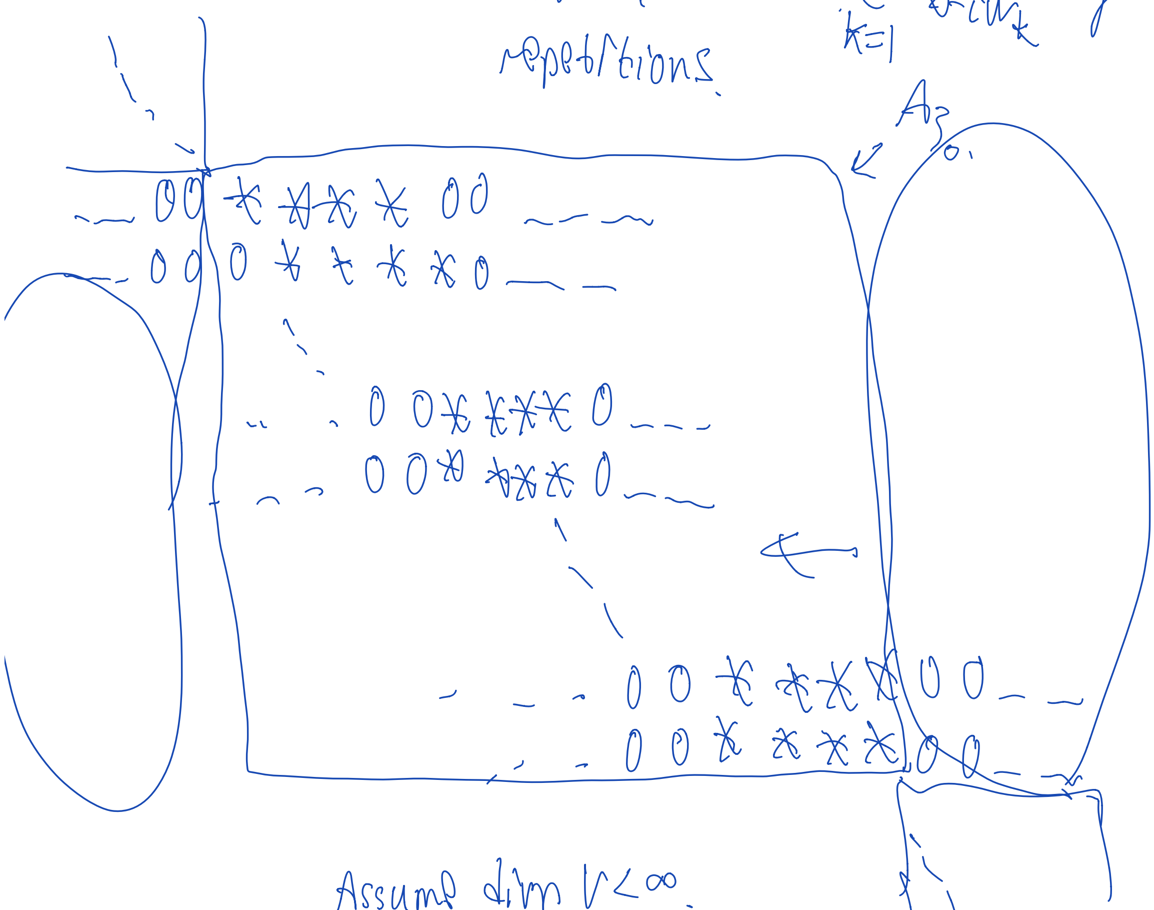
$$\zeta_0 + \frac{k}{\alpha} = \zeta + n$$

$k=0$	$\zeta = \zeta_0$	$n=0$
$k=1$	$\zeta = \zeta_0 + \tau$	$n=1$
$k=2$	$\zeta = \zeta_0 + 2\tau$	$n=2$
$\vdots$		
$k=t$	$\zeta = \zeta_0 + t\tau$	$n=t$
$k=t$	$\zeta = \zeta_0 + t\tau - 1$	$n=t+1$

$\zeta > 1$.

$N-1$ repetitions

$\sum_{k=1}^{N} \frac{d_k}{\dim \pi_k} = g$



Assume $\dim V < \infty$.

Prop (obv.) V: $v_1, v_2, \ldots, v_M \in V$ is a frame if and only if $\operatorname{span}\{v_1, \ldots, v_M\} = V$.

$\det(A_{3_0}) \neq 0 \Rightarrow$ frame.

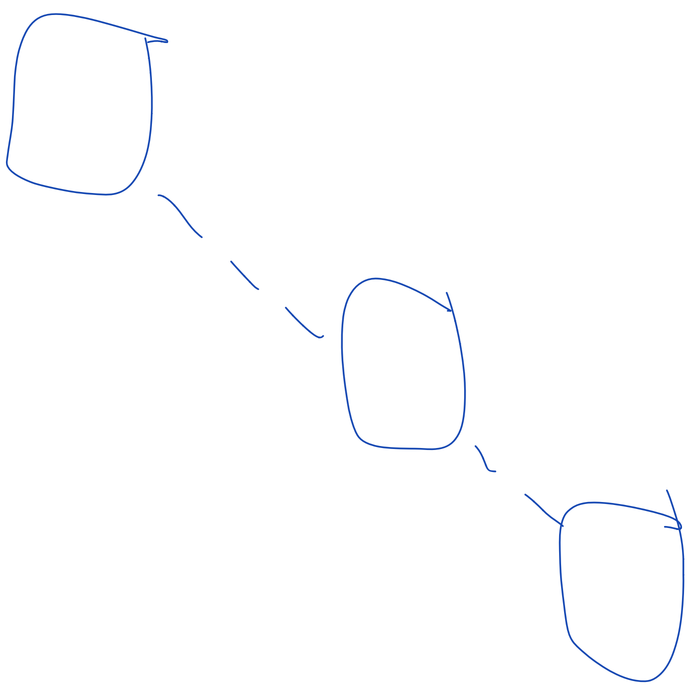
$\det\left(A_{3_0}(\alpha)\right) \neq 0$ for

all $\alpha \in (0, 1)$ ... $\in (0, 1)$

all $z_0 \in (0, \Sigma)$ $\alpha \in (0,1)$

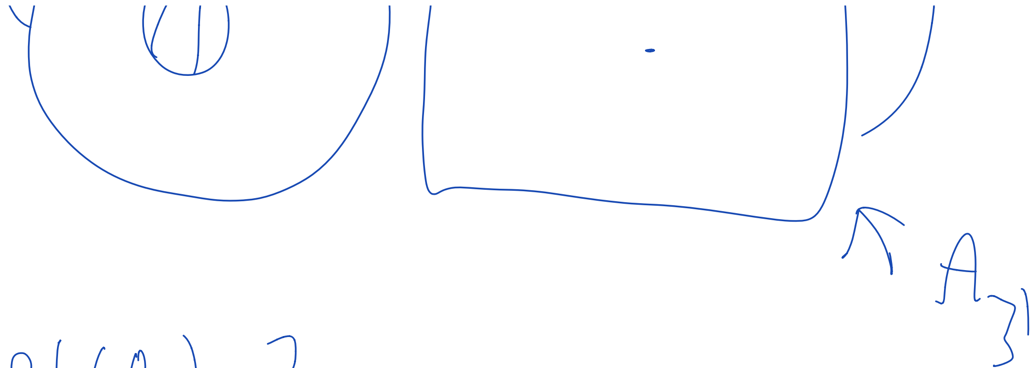
is too ambitious

Idea 2: $\underbrace{z \to z + \tau \bmod 1}$



$m_0(z) \neq 0$, $z > 0$.

$$\begin{pmatrix} x & x & x & x & 0 & \cdots & 0 & \cdots & & \\ & x & x & x & x & 0 & \cdots & & & \\ & & 0 & x & x & x & 0 & \cdots & & \\ & & (*) & & & & & & & \end{pmatrix} = B$$



$\det(B_z) = ?$

$\det B_z = \det A_z \cdot \underbrace{m_0(z) \cdot m_0(z+\tau) \cdots}_{\neq 0}$

$\alpha \notin \mathbb{Q} \Rightarrow \underset{\substack{\| \\ \alpha^{-1}}}{\tau} \notin \mathbb{Q}$

irrational motion is dense!

for any z $\exists$ k : $|\{z_0 + k\tau\} - z| < \varepsilon$.

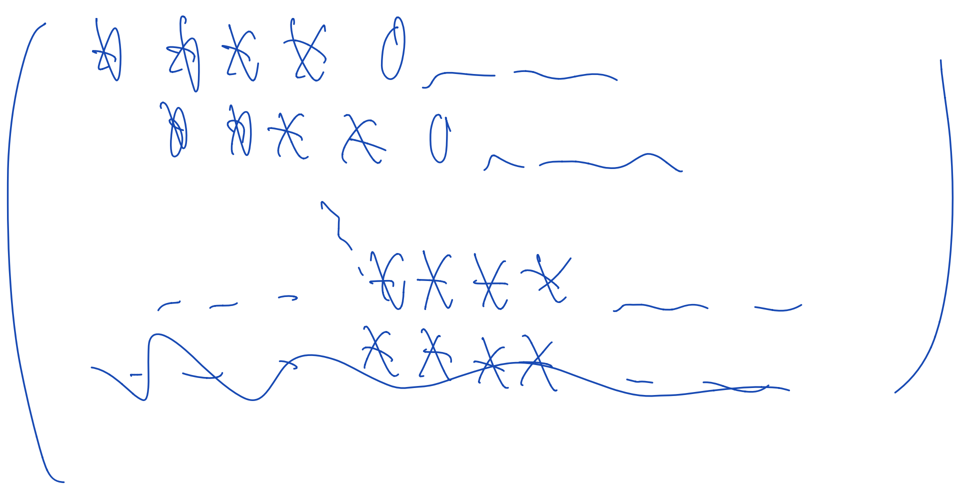
if $\det(A_z) \neq 0$

$(\det(A_z) \neq 0, \ z \in (z-\varepsilon, z+\varepsilon))$

instead of $\forall z \quad \det(A_z) \neq 0$

$\exists z \quad \det(A_z) \neq 0$

$(\det(A_z) \neq 0)$



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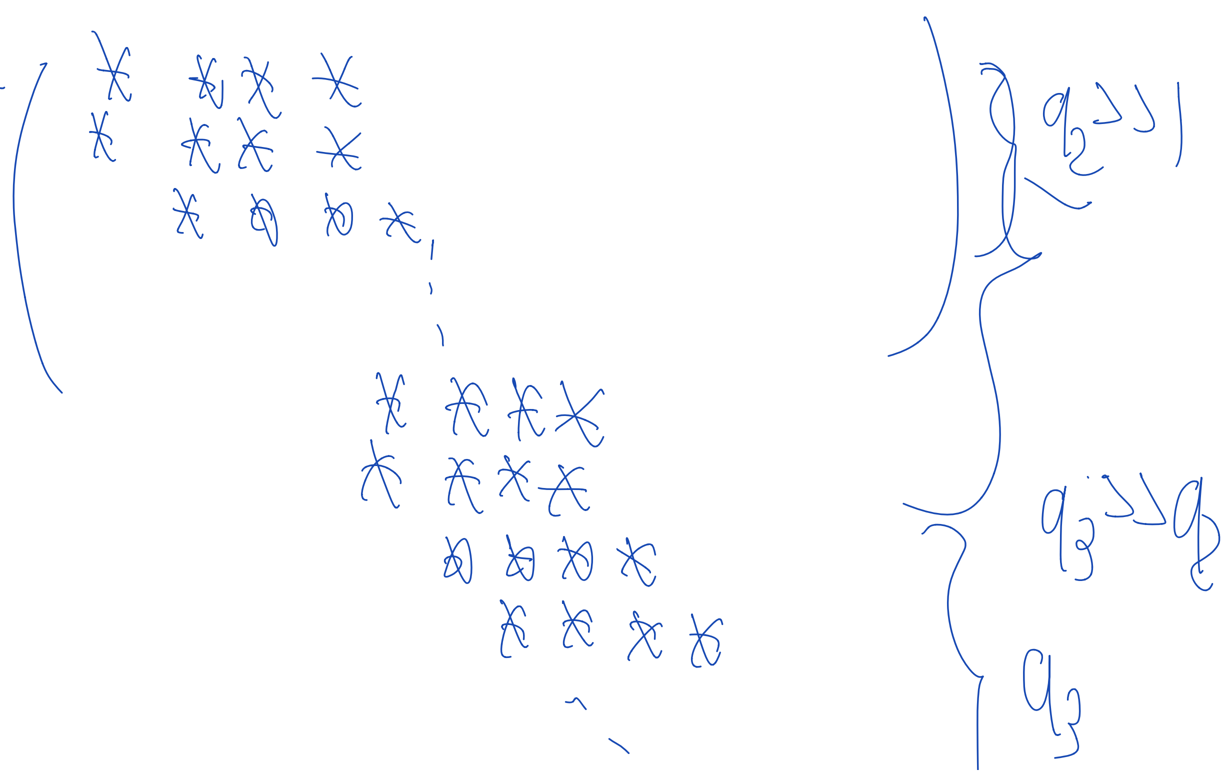
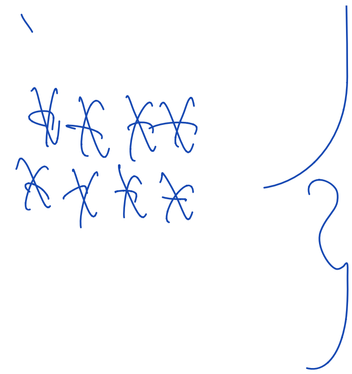




$q_{N-1} \gg q_{N-2}$

$N=2$,

$N>2$,

Prop.:
$$\frac{\sum_{s=0}^{N-1} m_s(\zeta)\, z^s}{\prod \left(1 - z\, e^{\frac{2\pi}{\alpha} w_k}\right)} = \sum \frac{a_k\, e^{2\pi i \zeta w_k}}{1 - z\, e^{\frac{2\pi}{\alpha} w_k}} \quad (\dagger)$$

Set $u_j = e^{\frac{2\pi}{\alpha} w_j}$

WLOG $\underline{\mathrm{Re}\, w_1 > \mathrm{Re}\, w_2} > \cdots > \mathrm{Re}\, w_N$

$\underline{|u_1| > |u_2| > \cdots > |u_N|}$

Compare residues at $z = u_j^{-1}$ in $(\dagger)$

$$\boxed{\sum_{s=0}^{N-1} m_s(\zeta)\, u_j^{-s}} = a_j\, \underline{u_j^{1-N}}\, e^{2\pi i \zeta w_j} \boxed{\prod_{\ell \neq j} (u_j - u_\ell)}$$

$$\left(1 - \frac{u_\ell}{u_j}\right) = u_j^{-1}\, (u_j - u_\ell)$$

$$
\left(\begin{array}{cccccccc}
* & * & * & * & 0 & \cdots & & 3 \\
* & * & * & * & 0 & \cdots & & 3-1 \\
* & * & * & * & 0 & \cdots & & 3-1+2 \\
 & & * & * & * & * & & \\
 & & * & * & * & * & & \\
 & & * & * & * & * & & \\
 & & & * & * & * & * & \\
 & & & * & * & * & * & \\
 & & & & & & * * * * & \\
 & & & & & & * * * * &
\end{array}\right) = F(\zeta)
$$

Q_2

$$0 = Q_1 << Q_2 << Q_3 << \cdots$$

double row # Q_2

$$<< Q_{n-1}$$

$$q_1 = 0$$

$$q_1 + q_2 + \cdots + q_{n-2} + q_{n-1} = k$$

$$F(\zeta)\begin{pmatrix} y_j^{k+n-1} \\ y_j^{k+n-2} \\ \vdots \\ y_j \\ 1 \end{pmatrix} \leftarrow m_0 \quad = \qquad m_0(\cdots)\, m_1(\cdots) \cdots m_{n-1}(\cdots)$$

$$= \boxed{a_j \cdot e^{2\pi i \xi w_j} \prod_{\ell \neq j} (u_j - u_\ell)} \times$$

$$X \begin{pmatrix} u_j^{K} \\ u_j^{K} \cdot e^{-2\pi i w_j} \\ u_j^{K-1} e^{-2\pi i w_j} e^{2\pi i \tau w_j} \\ \vdots \\ u_j^{K-q_2} e^{-2\pi i Q_2 w_j} e^{2\pi i q_2 \tau w_j} \\ u_j^{K-q_2} e^{-2\pi i (Q_2+1) w_j} e^{2\pi i q_2 \tau w_j} \\ \vdots \\ e^{-2\pi i Q_{N-1} w_j} e^{2\pi i K \tau w_j} \\ e^{-2\pi i (Q_{N-1}+1) w_j} e^{2\pi i K \tau w_j} \end{pmatrix} = V_j$$

v_i and w_j do not depend on z!

$W = (w_1, \ldots, w_N, \begin{pmatrix} 0_{N \times K} \\ I_{K \times K} \end{pmatrix})$

Put

$d(z) = \det(F(z) \cdot W)$

$$d(z) = e^{2\pi (w_1 + \ldots + w_N) z} \prod_{i=1}^{N} a_i \prod_{j \neq l} (u_j - u_l) \cdot \det(v_1, \ldots, v_N, F_{N+1}, \ldots, F_{K+N})$$

$\neq 0$

$f(z)$

columns of F.

Observation: $f(z)$ is a trig pol in z

$f(z) = \sum b_m e^{\lambda_m z}$ λ_m are distinct

a) if $f \not\equiv 0$ then f has at most fin many zeros

on $[0, \frac{1}{2})$

b) if any of $b_m \neq 0$

the $f \not\equiv 0$

$e^{2\pi i z u_j}$

$e^{2\pi i k u_j z}$ $e^{2\pi i z u_j}$

$$M(z) = (m_0(z), m_1(z), \dots, m_{N-1}(z))$$

$$M(z-1) - e^{-2\pi i w_1} M(z) =$$

$$= (\eta_0, \eta_1, \dots, \eta_{N-1})$$

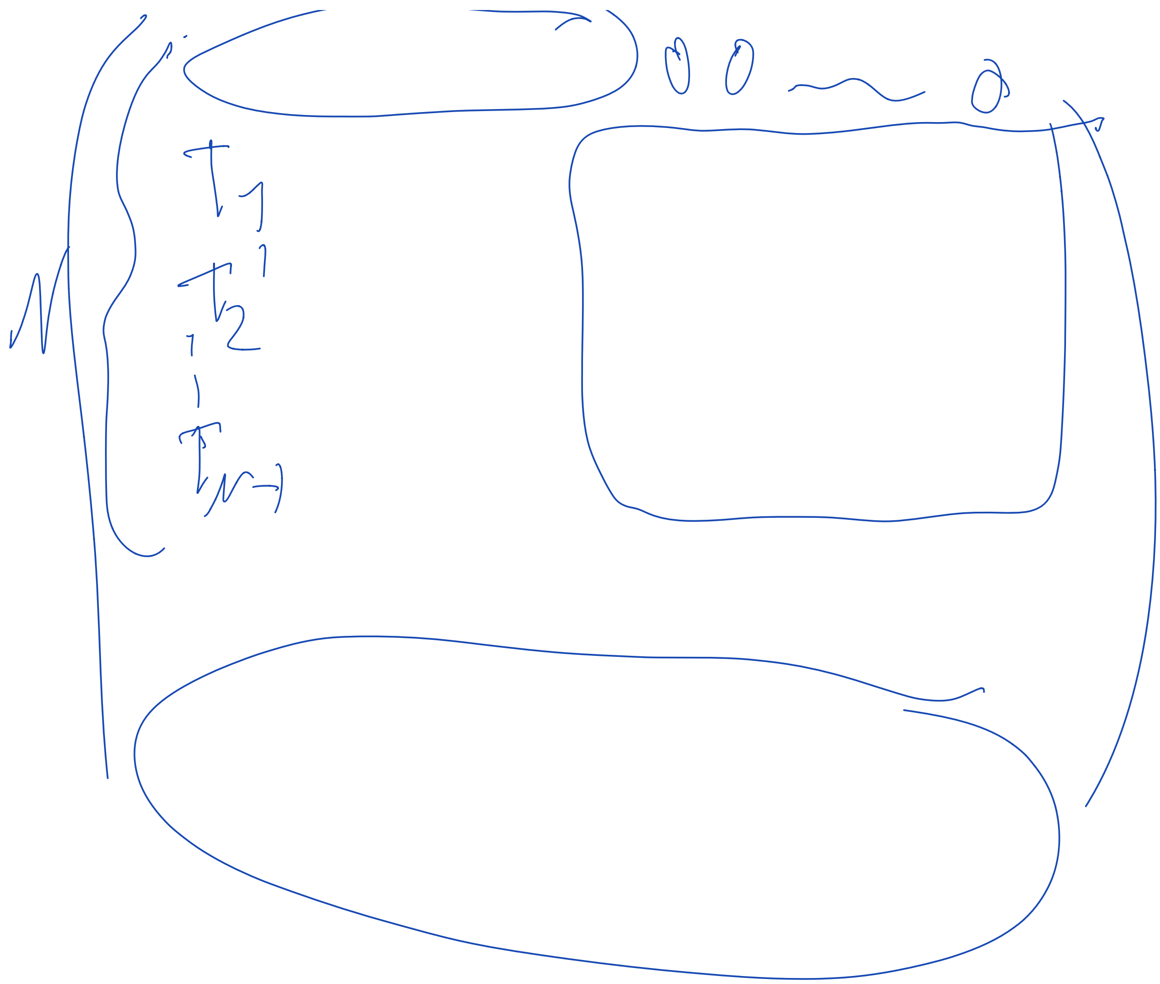
Claim: η_s do not have $e^{2\pi i z w_1}$

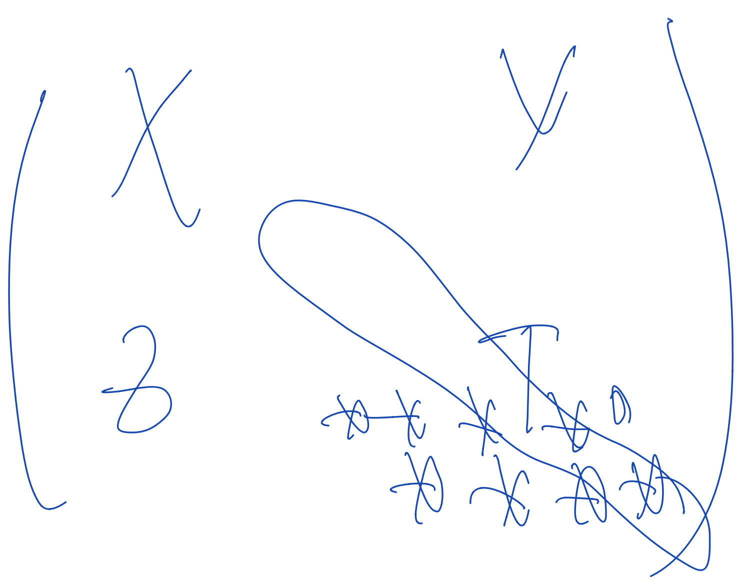
in each double row

$$R_s = (0 \dots 0 \; M(z - Q_s + (q_1 + \dots + q_s)\tau), 0 \dots)$$

$$R_s' = (0 \dots 0 \; M(z - Q_s - 1 + (q_1 + \dots + q_s)\tau), 0 \dots)$$

$$R_s' \rightsquigarrow T_s = R_s' - e^{-2\pi i w_1} R_s$$


n
T_1
T_2
T_{n-1}
0 0 ~ 0


X
Y
Z
T_0

Y, Z do not depend on ζ

$N \times N$ $\quad$ $K \times N$

Y $N \times K$ does not have $e^{2\pi i m_1 \zeta}$

T - lower-triangular matrix

$m_{N-1}(\zeta + \ldots)$ $\quad$ $e^{2\pi i \zeta m_1}$

coeff is $\neq 0$.

$$\begin{pmatrix} X & 0 \\ Z & T \end{pmatrix}$$

$\det(X)$ $\quad$ $\det(T)$

$$\ldots = \det(C_1)$$

Remains: $\det(X) \neq 0$

$$X = \begin{pmatrix} u_1^k & u_2^k & \cdots & u_N^k \\ 0 & u_2^k\left(e^{-2\pi i w_2} - e^{-2\pi i w_1}\right) & & u_N^k\left(e^{-2\pi i w_N} - e^{-2\pi i w_1}\right) \\ 0 & \vdots & & \\ \vdots & & & \\ 0 & & & \end{pmatrix}$$

$$u_2^{k-q_2}\left(e^{-2\pi i w_2} - e^{-2\pi i w_1}\right) e^{2\pi i w_2 (q_2 \tau - Q_2)}$$

$$\vdots$$

$$\left(e^{-2\pi i w_2} - e^{-2\pi i w_1}\right) e^{2\pi i w_2 \left(k\tau - Q_{k-1}\right)}$$

$$\in \left(0, \tfrac{1}{2}\right)$$

$$X = \begin{pmatrix} u_1^k & u_2^k & \cdots & u_N^k \\ 0 & \Theta(1)\, u_2^{k-a_2} & \cdots & \Theta(1)\, u_N^{k-a_2} \\ \vdots & \vdots & & \\ 0 & \Theta(1) & \cdots & \Theta(1) \end{pmatrix}$$

expand $\det(X) =$

$$= u_1^k \sum_{\sigma} \cdots$$

Claim: term with $\sigma = id$ dominates all the others

$(N-1)!$

$1 \ll q_2 \ll q_3 \ll q_4 \dots \ll q_{N-1}$

$|u_1| > |u_2| > \dots \quad |u_{N-1}|$,

$\alpha < 1 - \varepsilon$

$\alpha = \frac{p}{q} \qquad q > q(\varepsilon) \quad \checkmark$

$(p, q) = 1$