Gabor frames for rational functions

Lecture 3.

Hunting for positivity. Herglotz functions.

1. Dynamical system.
2. Frobenius matrices.
3. Bochkov's lemma.
4. Lax type transform

Th. Leg $g(x) = \sum_{k=1}^{N} \frac{a_k}{x - i\omega_k}$,

$a_k > 0$, $\omega_k > 0$.

Then $\mathcal{G}(g; \alpha, \beta)$ is a frame

$\iff \alpha\beta \le 1$.

<u>Proof.</u> Main Criterion + Dyn Syst.

$\beta = 1$.

$\boxed{\text{MC}}$ $\mathcal{G}(g; \alpha; 1)$ is a frame iff

$$\sum_{n \in \mathbb{Z}} \int_0^{1/\alpha} \Big| \sum_{s=0}^{N-1} G\big(\xi + n + \tfrac{s}{\alpha}\big)\, m_s(\xi) \Big|^2 d\xi \asymp \|G\|^2, \quad G \in L^2(\mathbb{R}).$$

<u>m_s is given.</u>

Zak type formula

are m_s

$$\sum_{k=1}^{N} \frac{a_k e^{2\pi\xi\omega_k}}{1 - z e^{2\pi/\alpha\, \omega_k}} = \frac{\sum_{s=0}^{N-1} m_s(\xi) z^s}{\prod_{k=1}^{N} \big(1 - z e^{2\pi/\alpha\, \omega_k}\big)}$$

We can control roots of $\sum_{s=0}^{N-1} m_s(\xi) z^s$

$m_s(\xi)$ – trigonometric polynomial.

$$m_s(\xi) = \sum_{k=1}^{N} a_k e^{2\pi i \xi w_k} \cdot A_{ks},$$

$$A_{ks} = (-1)^s \sum_{j_s \neq k} e^{\frac{2\pi}{\alpha}(w_{j_1} + \dots + w_{j_s})}.$$

$$1/\alpha \longrightarrow \underline{1}$$

$$\sum_{n \in \mathbb{Z}} \int_0^1 \Big| \sum_{s=0}^{N-1} G\left(\xi + n + \frac{s}{\alpha}\right) m_s(\xi) \Big|^2 d\xi$$

$$= \int_{\mathbb{R}} \Big| \sum_{s=0}^{N-1} G\left(\xi + \cancel{n} + \frac{s}{\alpha}\right) m_s(\{\xi\}) \Big|^2 d\xi$$

fractional part

$\xi \in [0,1)$

$n \in \mathbb{Z}$

$\xi + n \in \mathbb{R}.$

$$LG(\xi) = \sum_{s=0}^{N-1} G\left(\xi + \frac{s}{\alpha}\right) m_s(\{\xi\}).$$

$\stackrel{?}{=}$ $\|LG\| \geq \varepsilon \|G\|$, for some $\varepsilon > 0$.

$m_0(\xi) \neq 0$, or $m_{N-1}(\xi) \neq 0$.

$N=3$

$$LG(\xi) = G(\xi) + \underline{\tilde{m}_1(\{\xi\})}\, G(\xi + 1/\alpha) + \underline{\tilde{m}_2(\xi)}\, G(\xi + \tfrac{2}{\alpha}).$$

We can try to sole

$$LG = \underline{\underline{H}}(\xi), \quad \text{by } \underline{\underline{\text{iterations}}}$$

$$G(\xi) = H(\xi) - \tilde{m}_1(\{\xi\})\, \overline{G(\xi + \tfrac{1}{\alpha})} - \tilde{m}_2(\xi)\, \overline{G(\xi + \tfrac{2}{\alpha})}.$$

$$G(\xi) = H(\xi) - \tilde{m}_1(\{\xi\})\, H(\xi + 1/\alpha) - \tilde{m}_2(\{\xi\})\, H(\xi + 2/\alpha) + \tilde{m}_1(\{\xi\})\, \tilde{m}_2(\xi + 1/\alpha) + \ldots$$

$$G(\xi) = H(\xi) + \sum_{S=1}^{\infty} \bigcirc \, H(\xi + S/\alpha)$$

$$\sum m_{k_1} \cdot m_{k_2} \, m_{k_3} \cdots$$

$S = k_1 + k_2 + \cdots$

$k_\ell = 1$ or, $k_1 = 2$

combinatorics

Comb_S.

$N = 3$.

$$\mathrm{Comb}_S = (-1)^S \sum m_{k_1}(\xi)\, m_{k_2}\left(\xi + \frac{k_1}{\alpha}\right) m_{k_3}\left(\xi + \frac{k_1+k_2}{\alpha}\right) \cdots$$

$S = k_1 + \cdots + k_\ell$.

$k_j = 1$ or 2.

$$\begin{pmatrix} \mathrm{Comb}_{S+2} \\ \mathrm{Comb}_{S+1} \end{pmatrix} = \begin{pmatrix} m_1 & m_2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \mathrm{Comb}_{S+1} \\ \mathrm{Comb}_S \end{pmatrix}$$

depends on ξ

Start the proof.

Put $n = N-1$, $\quad LG(\xi) = \sum_{s=0}^{N-1} G(\xi + \frac{s}{\alpha}) m_s(\{\xi\})$.

$$m_n(\xi) = (-1)^n e^{\frac{2\pi i}{\alpha}(w_1 + \dots + w_n)} \sum_{k=1}^{N} a_k e^{2\pi i \xi w_k - \frac{2\pi i}{\alpha} w_k}.$$

$a_k > 0$, $\quad m_n(\xi) \neq 0$,

$PG(\xi) = \dfrac{LG(\xi)}{m_n(\{\xi\})}$, $\quad \|PG\| \overset{?}{\geq} \varepsilon \|G\|$.

$PG(\xi) = h(\xi)$, $\quad h$ is some $L^2(\mathbb{R})$

By iterations $\Updownarrow$

$$G(\xi + \frac{n}{\alpha}) = h(\xi) - \sum_{s=0}^{n-1} \frac{m_s(\xi)}{m_n(\xi)} G(\xi + \frac{s}{\alpha});$$

$$\Gamma(\xi) = \begin{pmatrix} G(\xi + \frac{n-1}{2}) \\ G(\xi + \frac{n-2}{2}) \\ \vdots \\ G(\xi + 1/2) \\ G(\xi) \end{pmatrix}$$

$$L^2(\mathbb{R}, \mathbb{C}^n).$$

How $\Gamma(\xi + 1/2)$ and $\Gamma(\xi)$.

$$H(\xi) = \begin{pmatrix} h(\xi) \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$\cap$$

$$L^2(\mathbb{R}, \mathbb{C}^n)$$

$$\Gamma(\xi + 1/2) = A_\xi \cdot \Gamma(\xi) + H(\xi)$$

matrix $(n \times n)$

$$A_\xi = \begin{pmatrix} \frac{m_0}{m_n}(\xi) & \cdots & \cdots & \frac{m_{n-1}}{m_n}(\xi) \\ 1 & & 0 & \\ & 1 & & \\ 0 & & \ddots & \\ & & 1 & 0 \end{pmatrix}.$$

$$\Gamma\left(\xi+\frac{1}{\alpha}\right)=\sum_{s=0}^{k} A_{\xi} A_{\xi+1/\alpha} \cdots A_{\xi+s/\alpha} H\left(\xi-\frac{s}{\alpha}\right)$$

$k \to \infty$

$$+\,\Gamma\left(\xi-\frac{k}{\alpha}\right)\cdot A_{\xi} \cdots A_{\xi-\frac{k}{\alpha}}.$$

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eigen values of A_ξ,

$$A=\begin{pmatrix} -b_{n-1} & \cdots & & -b_0 \\ 1 & & & \\ & \ddots & 0 & \\ 0 & & 1 & \end{pmatrix} \text{ — Frobenius matrix.}$$

$p(z)=z^n+b_{n-1}z^{n-1}+\dots+b_1 z+b_0$, (F-polynomial).

roots of p. eigenvalues of A.

eigenvectors $\begin{pmatrix} 1 \\ \lambda \\ \lambda^2 \\ \vdots \\ \lambda^{n-1} \end{pmatrix}$

$$A\begin{pmatrix} 1 \\ \vdots \\ \lambda^{n-1} \end{pmatrix}=\lambda\begin{pmatrix} 1 \\ \vdots \\ \lambda^{n-1} \end{pmatrix}.$$

$$\begin{pmatrix} -\frac{m_{n-1}}{m_n} & \cdots & -\frac{m_0}{m_n} \\ 1 & & 0 \\ & \ddots & \\ 0 & & 1 \quad 0 \end{pmatrix}$$

$$m_n(\xi) p(z) = m_n(\xi) z^n + m_{n-1}(\xi) z^{n-1} + \dots + m_1(\xi) z + m_0(\xi).$$

Zak type transform formula

$$\sum_{k=1}^{N} \frac{a_k e^{2\pi i \xi w_k}}{1 - z e^{2\pi i \xi/\alpha w_k}} = \frac{p(z) \cdot m_n(\xi)}{\prod(\dots)}$$

$a_k > 0$, $w_k > 0$. roots of p. $\in [0,1]$

and interlaced with $\{e^{-\frac{2\pi}{\alpha} w_k}\}$ poles

$p(z)$, $\lambda_n < \lambda_{n-1} < \dots < \lambda_1$

Put, $\mu_k = e^{-\frac{2\pi}{\alpha} w_k}$, $\mu_n < \lambda_n < \mu_{n-1} < \dots$

$< \lambda_2 < \mu_2 < \lambda_1 < \mu_1 \dots$

μ_k - fixed, $\lambda_k = \lambda_k(\xi)$,

$$\mu_{k+1} < \lambda_k(\xi) < \mu_k.$$

Lemma (Frost matri I. Bochnov).

$$0 < \mu_{n+1} < \mu_n < \dots < \mu_1 < 1.$$

$$p(x) = (x-\lambda_1)\dots(x-\lambda_n)$$

$\{\lambda_k\}$ interlacing with μ_k

$\Rightarrow \exists\ C > 0,\ q < 1.$

$p_1, \dots p_m$ of such type.

$$\| A_{p_1} \dots A_{p_m} \| \leq C \cdot q^m.$$

Proof $\quad p(x) = (x-\lambda_1)(x-\lambda_2)\dots(x-\lambda_n).$

$$\mu_{k+1} \leq \lambda_k \leq \mu_k.$$

① Trick. $\lambda_k = a\,\mu_{k+1} + b\,\mu_k$

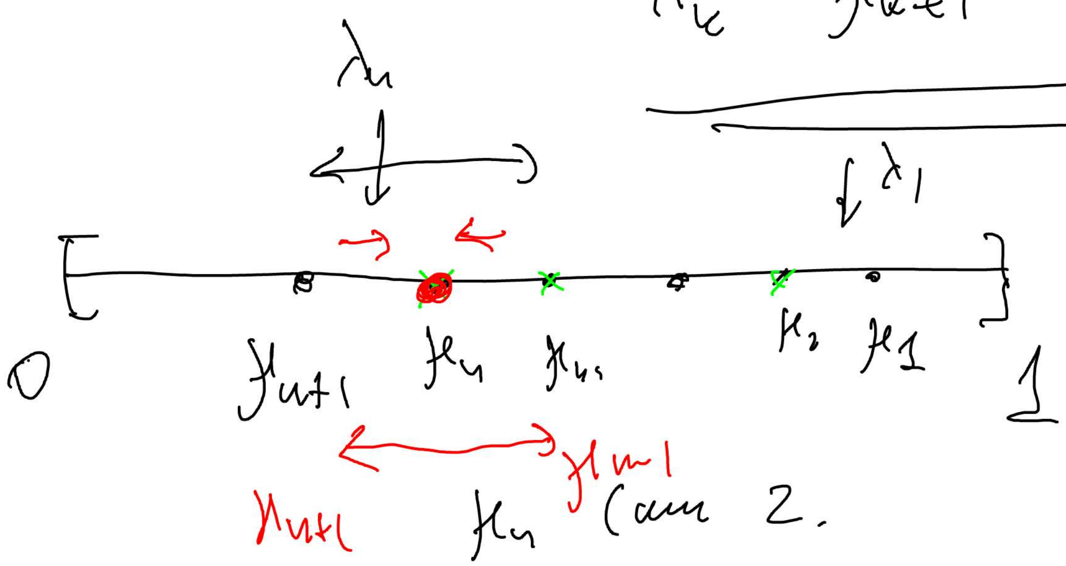
$0 \le a, b, \quad a+b=1.$

$$A_p = a A_{\lambda_k \to \mu_{k+1}} + b A_{\lambda_k \to \mu_k}.$$

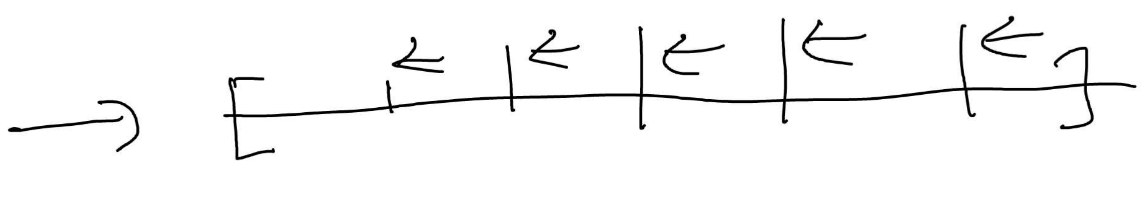
$A_p \to$ convex linear combination A

$$A_p = \text{convex comb}\left(A_{p^*} \right)$$

p^* - is a polynomial $\lambda_k = \mu_{k+1}$ or μ_k.

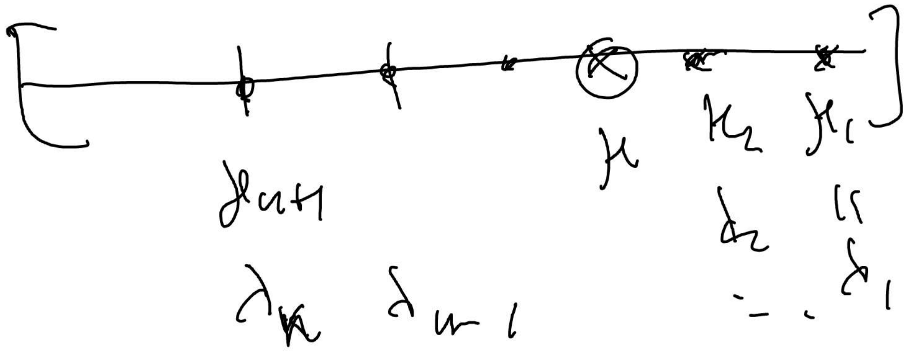


Case 2.



$n+1$ dof of polyn

p_{k1} $\quad 1 \le k \le n+1$



$$\underline{\underline{p_\ell(z)}} = \prod_{\substack{k=1 \\ k\neq \ell}}^{n+1} (z-\mu_k)$$

$n+1$ FM.

We have to construct norm.

A_{p_ℓ} is contractive with respect to <u>this norm</u>.

$1 \le \ell \le n+1$

roots of p_ℓ $\quad \{\mu_k\}_{k=1}^{n+1} \setminus \{\mu_\ell\}$.

Constr. of the norm. A_{p_ℓ}.

$$A_{p_\ell}^T = \begin{pmatrix} * & 1 & & 0 \\ * & & 1 & \\ \vdots & 0 & & \ddots \\ * & & & 1 \end{pmatrix} \qquad \underline{\underline{p_\ell(z)}}$$

$$\|A_{p_e}\|_* \le q < 1$$

column

$$A_{p_e}^T \cdot Q = Q^*$$

$$Q = \begin{pmatrix} q_0 \\ \vdots \\ q_{n-1} \end{pmatrix}.$$

$$Q \sim \sum_{k=0}^{n} q_k z^k$$

$$Q^*(z) = z \cdot Q(z) \bmod p_e(z).$$

Polynomial ~~de~~ well defined

$$\{Q(\mathcal{H}_e)\}_{e=1}^{n+1}$$

$$Q^*(\mathcal{H}_s) = \mathcal{H}_s Q(\mathcal{H}_s), \quad s \neq l$$

$\boxed{\text{Norm}}$ Q of degree $n-1$ (n-dimensional linear space).

$$Q(\mathcal{H}_s), \quad s = 1, \dots n+1.$$

linear transform

$n+1$ linear functionals.

$\exists\, a_k,$ s.t. $\sum_{k=1}^{n+1} a_k Q(\mu_k) = 0.$

$\forall$ polynomial Q of degree $\leq n-1$

Norm. $\|Q\|_{*1} = \sum_{k=1}^{n+1} |a_k Q(\mu_k)|,$

our norm. on $\mathbb{C}^n,$ $\begin{pmatrix} \vdots \end{pmatrix} \to$ polynomial,

l-fixed

$$\boxed{\|A_{pe}^T Q\|_* \leq \mu_1 \cdot \|Q\|_{*}.} \quad \mu_1 < 1$$

$l = 1, 2, \dots n+1,$

$\mu_k < \mu_1$

$$\sum_{k \neq l} |a_k \mu_k Q(\mu_k)| + |\underline{a_l} \hat{Q}(\mu_l)| =$$

$$= \sum_{k \neq l} |a_k \mu_k Q(\mu_k)| + \Big|\sum_{k \neq l} a_k \mu_k Q(\mu_k)\Big|$$

$$= \mu_1 \Big(\sum_{k \neq l} \Big| a_k \overbrace{\frac{\mu_k}{\mu_1}}^{\leq 1} Q(\mu_k) \Big| + \Big| \sum_{k \neq l} a_k \overbrace{\frac{\mu_k}{\mu_1}}^{\leq 1} Q(\mu_k) \Big| \Big)$$

?

$$\leq \mu_1 \cdot \sum_{k} | a_k Q(\mu_k) | =$$

$$= \mu_1 \Big(\sum_{k \neq l} | a_k Q(\mu_k) | + \Big| \sum_{k \neq l} a_k Q(\mu_k) \Big| \Big).$$

Lemma. $x_k \in \mathbb{R}$, $s_k \in [0,1]$.

$$\sum_{k \neq l} |s_k x_k| + \Big| \sum_{k \neq l} s_k x_k \Big| \leq$$

$$\leq \sum_{k \neq l} |x_k| + \Big| \sum_{k \neq l} x_k \Big|.$$

Proof.

$$\left|\sum_{k\neq l} s_k x_k\right| - \left|\sum_{k\neq l} x_k\right| \qquad s_k\in[0,1]$$

$$\leq \left|\sum_{k\neq l} x_k(1-s_k)\right| =$$

$$\leq \sum_{k\neq l} |x_k|\cdot(1-s_k)$$

$$= \sum_{k\neq l}|x_k| - \sum_{k\neq l}|x_k\cdot s_k|$$

Hengl(?)