Gabor analysis for rational functions

joint project with

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Lecture 1. Introduction to Gabor analysis. New results.

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0. Fourier analysis vs Time-frequency analysis
1. Time-frequency shifts. Short time Fourier transform.
2. Reconstruction. Sampling. Frames. Overview of general theory.
3. Gabor frames. General results. Main questions.
4. Gabor frame for concrete windows.
5. New results and approaches.

FA. $L^2[0,1]$, $f(t) = \sum_{n\in\mathbb{Z}} a_n e^{2\pi i n t}$,

$f \in L^2[0,1]$, $a_n \in \ell^2$, $\sum_{n\in\mathbb{Z}} |a_n|^2 < \infty$

$$\|f\|^2 = \int_0^{2\pi} |f(t)|^2 dt = \sum_{n\in\mathbb{Z}} |a_n|^2.$$

Non-classical $\Lambda \subset \mathbb{R}$ - discrete

$$f(t) = \sum_{\lambda\in\Lambda} a_\lambda e^{2\pi i \lambda t}.$$

$L^2(\mathbb{R})$ Fourier transform

$$\hat{f}(\omega) = \int_{\mathbb{R}} f(t) e^{-2\pi i \omega t} dt,$$

$\hat{f}(\omega)$ - amplitude of function f w r freq. ω.

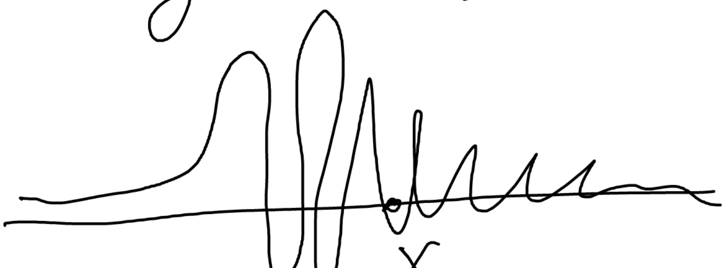
g - window $g \in L^2(\mathbb{R})$.

0

Time-frequency shifts of g.

$x, \omega \in \mathbb{R}$.

$$g_{x,\omega}(t) = e^{2\pi i \omega t}\, g(t-x),$$



x

Re $e^{2\pi i \omega t} e^{-\pi(t-x)^2}$.

Gabor samples.

$f \in L^2(\mathbb{R})$, reconstruct via

$$\{(f, g_{x,\omega})_{L^2}\}_{(x,\omega)\in\Lambda} \quad (\mathrm{I})$$

$\Lambda \subset \mathbb{R}^2$

Short-time Fourier transform

$$V_g f(x,\omega) = \int_{\mathbb{R}} f(t)\, e^{-2\pi i \omega t}\, \overline{g(t-x)}\, dt, \quad (\mathrm{II})$$

$(f, g_{x,\omega}) \sim$ amplitude of function f w.r.t. frequence ω near the time x.

$\sim$ Musical score

$V_g f(x,\omega) \in L^2(\mathbb{R}^2)$.

It is suffice to consider only $(x,\omega) \in \Lambda \subset \mathbb{R}^2$.

Example. $g(t) = \chi_{[0,1]}(t) = \begin{cases} 1, & t \in [0,1] \\ 0 & \text{otherwise.} \end{cases}$

$\Lambda = \mathbb{Z} \times \mathbb{Z}$.

$L^2(0,1)$, $\quad \{g_{m,n}(t)\}_{m,n \in \mathbb{Z}}$ — orthogonal basis in $L^2(\mathbb{R})$.

Gabor system.

Fix $g \in L^2(\mathbb{R})$, Λ - discrete subset of $\mathbb{R}^2$.

$$\{g_{x,\omega}\}_{(x,\omega)\in\Lambda} \subset L^2(\mathbb{R}).$$

$$f(t) = \sum_{(x,\omega)\in\Lambda} c_{(x,\omega)}\, g_{x,\omega}(t), \quad (1).$$

$$f(t) = \sum_{(x,\omega)\in\Lambda} (f, g_{x,\omega}) \cdot h_{x,\omega}. \quad (2).$$

Hilbert space $\{e_n\}$.

$f \sim (f, e_n)$,

Completeness property $(f, e_n) = 0 \Rightarrow f \equiv 0$.

Frame property. $\{e_n\}$ - frame.

$$A\|f\|^2 \leq \sum_{n\in\mathbb{N}} |(f,e_n)|^2 \leq B\|f\|^2$$

$A, B > 0$,

If $\{e_n\}$ forms a frame.

$\exists$ canonical way. $f = \sum_{n \in N} c_n e_n$.

For Gabor analysis

<u>Def.</u> $\{g_{x,\omega}\}_{(x,\omega) \in \Lambda}$ - is a Gabor frame.

if $\{g_{x,\omega}\}_{(x,\omega) \in \Lambda}$ forms a fram in $L^2(\mathbb{R})$!

$$\Lambda = \alpha \mathbb{Z} \times \beta \mathbb{Z}, \quad \alpha, \beta > 0.$$

<u>Th.</u> ~~Let~~ $\{g_{\alpha m, \beta n}\}_{m,n \in \mathbb{Z}}$ - Gabor frame.

Then. there exists dual window γ.

$f \in L^2(\mathbb{R})$, $\quad f = \sum_{m,n \in \mathbb{Z}} (f, \gamma_{\alpha m, \beta n}) \, g_{\alpha m, \beta n}$

$$= \sum_{m} (f, g_{\alpha m, \beta n}) \cdot \gamma_{\alpha m, \beta n}.$$

Main questions of Gabor analysis.

1. $G(g;\alpha,\beta) = \{g_{\alpha m,\beta n}\}_{m,n\in\mathbb{Z}}$.

Fix g. describe a frame set for g

$\mathcal{F}(g) = \{(\alpha,\beta) : G(g,\alpha,\beta)$ - forms a frame in $L^2(\mathbb{R})\}$.

2. What happens with constants.

$\underline{\underline{(A,B)}} = (A_{\alpha\beta}, B_{\alpha,\beta})$, $\underline{\underline{B_{\alpha,\beta}}} \le C$.

What is known in general.

$\underline{\underline{\text{Th.}}}$ If $\alpha\beta > 1$, then $G(g,\alpha,\beta)$ is not a frame.

$\underline{\underline{\text{Th.}}}$ If $\alpha+\beta < \varepsilon$, $\varepsilon = \varepsilon(g) > 0$, then $G(g,\alpha,\beta)$ $\underline{\underline{\text{forms a frame}}}$.

Balian-Low theorem

Th. Let g is smooth and decaying.

$$\int_{\mathbb{R}} |t|^2 |g(t)|^2 \, dt + \int_{\mathbb{R}} |\omega|^2 |\hat{g}(\omega)|^2 \, d\omega < \infty$$

and $\alpha\beta = 1 \Rightarrow G(g, \alpha, \beta)$ is not a frame.

Daubechies conjecture.

If $g > 0$ and $\hat{g} > 0$,

then $G(g, \alpha, \beta)$ is a frame

if $\alpha\beta < 1$. (1988)

Disproved by Janssen (1996)

Positivity idea

Concrete windows

2011   6 functions.

Remark. 1. $G(g,\alpha,\beta) = G(\hat{g}, \beta, \alpha)$

2. $g(t) \rightsquigarrow g(ct)$. Dilations.

1. Gaussian, $g(t) = e^{-\pi t^2}$,

$G(g,\alpha,\beta)$ is a frame iff $\alpha\beta < 1$.

Lyubarskii, Seip, 1992 or earlier

2. Hyperbolic secant

$g(t) = \dfrac{1}{e^t + e^{-t}}$, Janssen - Strohmer 2002.

$G(g,\alpha,\beta)$ is a frame iff $\alpha\beta < 1$.

3. One sided exponential. Janssen 1996.

$g(t) = e^{-t}\chi_{[0,\infty)}(t)$, $\Leftrightarrow \alpha\beta \le 1$.

4. Two sided exponential function

$e^{-|t|}$, $G(g,\alpha,\beta)$ is a frame $\iff \alpha\beta < 1$.

2011 (...) Totally positive functions.

Def. $g:\mathbb{R}\to\mathbb{R}$, $x_1 < x_2 \dots < x_N$, $y_1 < y_2 \dots < y_N$.

$$\det[g(x_j - y_k)]_{1\le j,k\le N} \geq 0.$$

1951 (Schoenberg).

g is ~~totaly~~ positive

$$\iff \hat{g}(\xi) = C e^{-a\xi^2} \cdot e^{ib\xi} \prod_{k=1} \frac{e^{i\delta_k\xi}}{1+i\delta_k\xi},$$

Moras

$\delta_k \in \mathbb{R}$, $\sum_k \delta_k^2 < \infty$

$a = 0$

$e^{-\pi x^2}$ - TP.

$\widehat{e^{-x}} = \frac{1}{\xi + i}$ - TP.

$\widehat{e^{-|x|}} = \frac{1}{1+\xi^2}$ - TP. $= \frac{1}{(\xi+i)(\xi-i)}$

$\frac{1}{e^t + e^{-t}}$ - TP.

(2011) (Gröchenig, Stöckler 2011)

$$\hat{g}(\xi) = \prod_{k=1}^{M} \frac{1}{1 + i\delta_k \xi}, \quad M > 1$$

$\mathcal{G}(g, \alpha, \beta)$ is a frame $\Leftrightarrow \alpha\beta < 1$.

(2017) (Gröchenig, Romero, Stöckler).

$$\hat{g}(\xi) = e^{-a\xi^2} \cdot \prod_{k=1}^{M} \frac{1}{1 + i\delta_k \xi},$$

$\mathcal{G}(g, \alpha, \beta)$ is a frame $\Leftrightarrow \alpha\beta < 1$.

2016 Xin-Rong, Qiyu (first result Gaussian).

$g(t) = \chi_{[0,1]}(t)$.

frame set is complicated

$g(x) = x e^{-\pi x^2}$, if $\alpha\beta = \frac{n-1}{n}$ $(n = 2, 3 \ldots$

$= \frac{1}{2}, \frac{2}{3}, \frac{3}{4} \ldots$

$\Rightarrow G(g, \alpha, \beta)$ is not a frame

Lyubarskii, Nes 2006

Conj $g(t) = t e^{-\pi t^2}$, $G(g, \alpha, \beta)$ is a frame iff

$\alpha\beta < 1$, $\alpha\beta \neq \frac{n-1}{n}$.

α, β, $\alpha\beta$

New results and approaches

Th.1. (A. Kulikov, Y. Lyubarskii + YB)

$$g(t) = \sum_{k=1}^{N} \frac{a_k}{t - i w_k}, \qquad \begin{array}{l} w_k > 0 \\ a_k > 0. \end{array}$$

$G(g, \alpha, \beta)$ is a frame iff. $\alpha\beta \le 1$.

Th.2. (—)

$$g(t) = \sum_{k=1}^{N} \frac{a_k}{t - i w_k}, \qquad \begin{array}{l} a_k, w_k \in \mathbb{C} \\ w_k \in i\mathbb{R} \end{array}$$

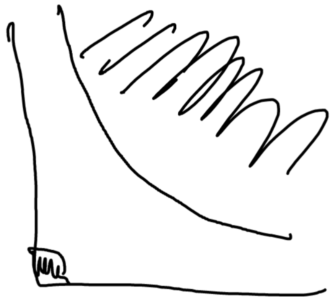
$$\sum_{k=1}^{N} a_k e^{2\pi \xi w_k} \ne 0, \ \xi > 0, \qquad \operatorname{Re} w_k \ne \operatorname{Re} w_\ell, \ k \ne \ell.$$

If $\alpha\beta < 1$ and $\alpha\beta \notin \mathbb{Q}$

$\Rightarrow G(g, \alpha, \beta)$ is a frame.

Daubechies conjecture is true for rational function and irrationalities.

Th. 3. (Near critical hyperbola)



$$Z(z,\xi) = \sum_{k=1}^{N} \frac{a_k e^{2\pi\xi w_k}}{1 - z e^{2\pi/\alpha\, w_k}},$$

Let $\mathrm{Re}\, w_k > 0$, $\mathrm{Re}\, Z(e^{2\pi i t}, \xi) > 0$,

then there exists $\alpha_0 < 1$ s.t.

$G(g,\alpha,1)$ is a frame for

$\alpha_0 \le \alpha < 1$,

$\beta = 1$, If we rescale g.

~~Th 2.4~~

Corollary of Th. 2 Under the Ass. of Th 2..) $\forall \delta > 0$, $\exists\, q_0 = q(\delta)$.

s.t. $\alpha = p/q$, $q > q_0$

$p/q < 1 - \delta$

$\Longrightarrow G(g,\alpha,1)$ a frame.

Th. 2 + Th. 3.

$\beta = 1$ fix

finite exceptional set

Th. 4. Near origin. $g(t) = \sum_{k=1}^{N} \frac{a_k}{t - i\omega_k}$,

$\mathrm{Re}\,\omega_k \neq \mathrm{Re}\,\omega_\ell$
$k \neq \ell$

if. $\alpha\beta \le \frac{1}{N}$,

$\Longrightarrow G(g,\alpha,\beta)$ is a frame

$\exists\, g$ with $\mathrm{Re}\,\omega_k \neq \mathrm{Re}\,\omega_\ell$, s.t.

$G(g,\alpha,\beta)$ is not a frame

$\alpha\beta = \frac{1}{N-1}$

(1) + $\frac{1}{x+i}$ Irregular sampling.

$(\alpha\mathbb{Z}\times\beta\mathbb{Z},\ \Lambda\times M)$.

iff condition.

(2). $g(x) = \frac{a_1}{x-iw_1} + \frac{a_2}{(x-iw_2)}$.

complete characterization of frame set

KIS (sandwhich is complicated)

(3). $\frac{1}{1+x^2}$.) asymptotics of lower bound.

(known only for Gaussian).

(4) Multiple poles. ~~$\sum$~~ $\frac{P(t)}{(t+i)^n}$

$\rightarrow$ frame set. (Toeplitz approach).

First idea.

$$\frac{1}{x+i}$$ another proof