Almost harmonic Maass forms
and Kac-Wakimoto characters

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Kac-Wakimoto characters

Questions of Kac:

Investigate

- **automorphic properties**
- **asymptotic behaviors**

of characters for affine Lie superalgebras $s\ell(m|n)^{\wedge}$.

Kac-Wakimoto characters

History

- **Monstrous Moonshine:**

  $j(\tau) = q^{-1} + 744 + 196884q + \cdots \quad \longleftrightarrow \quad$ Monster group $\mathbb{M}$.

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  Rogers-Ramanujan identites $\quad \longleftrightarrow \quad A_1^{(1)}$ characters.

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- etc.

Kac-Wakimoto characters

A character formula for $n = 1$:

Proposition (Kac, Wakimoto '01)

For $m \geq 2$, $s \in \mathbb{Z}$, we have

$$tr_s(m,1) = 2q^{\frac{m-2}{24} - \frac{s}{2}} \cdot \frac{\eta^2(2\tau)}{\eta^{m+2}(\tau)} \sum_{\vec{k} \in \mathbb{Z}^{m-1}} \frac{q^{\frac{1}{2}\sum_i k_i(k_i+1)}}{1 + q^{-s+\sum_i k_i}}$$

$q = q_\tau := e^{2\pi i \tau}, \tau \in \mathbb{H}, \quad \eta(\tau) := q^{\frac{1}{24}} \prod_{k=1}^{\infty}(1 - q^k)$

Kac-Wakimoto characters

A generating function for $n \geq 1$:

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For $m \geq 2, n \geq 1,$ *we have*

$$\sum_{s\in\mathbb{Z}} \mathrm{tr}_s(m,n)\zeta^s = \prod_{n=1}^{\infty}(1-q^n)\prod_{k=1}^{\infty}\frac{\left[\left(1+\zeta q^{k-\frac{1}{2}}\right)\left(1+\zeta^{-1}q^{k-\frac{1}{2}}\right)\right]^m}{\left[\left(1-\zeta q^{k-\frac{1}{2}}\right)\left(1-\zeta^{-1}q^{k-\frac{1}{2}}\right)\right]^n}$$

$\zeta = e^{2\pi i z}$

Kac-Wakimoto characters

I. Automorphic properties •$n = 1$ [Bringmann-Ono '09, F '11]

•$n > 1$ [Bringmann-F '12]

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II. Asymptotic behaviors •$n = 1$ [Bringmann-F ‘11]
[Bringmann-Mahlburg ‘11]

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Kac-Wakimoto characters

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Kac-Wakimoto characters

Refined question:

Relation between K-W characters and mock theta functions?

Mock theta functions

Ramanujan's Mock θ-functions

Example.

$$f(q) = \sum_{n\geq 0} \frac{q^{n^2}}{(-q;q)_n^2}$$

$(a;q)_n := (1-a)(1-aq)\dots(1-aq^{n-1})$

Mock theta functions

Remark.

$$f(q) = 2q^{1/24}\eta^{-1}(\tau) \cdot \sum_{n\in\mathbb{Z}} \frac{(-1)^n q^{\frac{3n^2}{2}+\frac{n}{2}}}{1+q^n}$$

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$$tr_s(m,1) = 2q^{\frac{m-2}{24}-\frac{s}{2}} \cdot \frac{\eta^2(2\tau)}{\eta^{m+2}(\tau)} \quad \cdot \sum_{\vec{k}\in\mathbb{Z}^{m-1}} \frac{q^{\frac{1}{2}\sum_i k_i(k_i+1)}}{1+q^{-s+\sum_i k_i}}$$

Zwegers's work

S. Zwegers '02: The mock theta functions may be "completed" to transform as non-holomorphic modular forms.

Zwegers's work

Let $\tau \in \mathbb{H}$, $u, v \in \mathbb{C} \setminus (\mathbb{Z}\tau + \mathbb{Z})$, $z := e^{2\pi i u}$, $w := e^{2\pi i v}$, $q = e^{2\pi i \tau}$.

Appell-Lerch sums

$$\mu(u, v; \tau) := \frac{z^{1/2}}{\vartheta(v; \tau)} \cdot \sum_{n \in \mathbb{Z}} \frac{(-w)^n q^{n(n+1)/2}}{1 - zq^n}$$

$$\vartheta(v; \tau) := \sum_{\nu \in \mathbb{Z} + \frac{1}{2}} e^{\pi i \nu} w^{\nu} q^{\nu^2/2}.$$

Zwegers's work

Zwegers's completion:

$$\widehat{\mu}(u, v; \tau) := \mu(u, v; \tau) + \frac{i}{2} R(u - v; \tau)$$

$$R(u; \tau) := \sum_{\nu \in \mathbb{Z} + \frac{1}{2}} (-1)^{\nu - \frac{1}{2}} \left\{ \mathrm{sgn}(\nu) - E\left((\nu + c)\sqrt{2\mathrm{Im}(\tau)} \right) \right\} e^{-2\pi i \nu u} q^{-\nu^2/2}$$

$$E(x) := 2 \int_0^x e^{-\pi u^2} du = \mathrm{sgn}(x)(1 - \beta(x^2))$$

$$\beta(x) := \int_x^\infty u^{-\frac{1}{2}} e^{-\pi u} du$$

Zwegers's work

Theorem (Zwegers '02)

Assuming the notation and hypotheses above,

i. $\widehat{\mu}(u, v; \tau) = \widehat{\mu}(v, u; \tau)$,

ii. $\widehat{\mu}(u+1, v; \tau) = z^{-1}wq^{-\frac{1}{2}}\widehat{\mu}(u+\tau, v; \tau) = -\widehat{\mu}(u, v; \tau)$,

iii. $\zeta_8\widehat{\mu}(u, v; \tau+1) = (-\tau/i)^{-\frac{1}{2}}e^{\pi i(u-v)^2/\tau}\widehat{\mu}\left(\frac{u}{\tau}, \frac{v}{\tau}; -\frac{1}{\tau}\right) = \widehat{\mu}(u, v; \tau)$,

iv. $\widehat{\mu}\left(\frac{u}{\gamma\tau+\delta}, \frac{v}{\gamma\tau+\delta}; \frac{\alpha\tau+\beta}{\gamma\tau+\delta}\right) = \chi(A)^{-3}(\gamma\tau+\delta)^{\frac{1}{2}}e^{-\pi i\gamma(u-v)^2/(\gamma\tau+\delta)} \cdot \widehat{\mu}(u, v; \tau)$.

Harmonic Maass forms

Definition (Bruinier-Funke, '04)

A *harmonic Maass form* of weight $k \in \frac{1}{2}\mathbb{Z}$ on $\Gamma_0(4N)$ is a smooth $f : \mathbb{H} \to \mathbb{C}$ satisfying

- $\forall A \in \Gamma_0(4N)$, $\tau \in \mathbb{H}$,

$$f(A\tau) = \begin{cases} \left(\frac{c}{d}\right)^{2k} \epsilon_d^{-2k} \, (c\tau + d)^k f(\tau), & k \in \frac{1}{2}\mathbb{Z} - \mathbb{Z} \\ (c\tau + d)^k f(\tau), & k \in \mathbb{Z} \end{cases}$$

- $\Delta_k f = 0$, where $\Delta_k := -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + iky \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$

- f has at most linear exponential growth at all cusps.

$$\epsilon_d := \begin{cases} 1, & d \equiv 1 \mod 4, \\ i, & d \equiv 3 \mod 4. \end{cases} \qquad \tau := x + iy \in \mathbb{H}$$

Harmonic Maass forms

Decomposition: $f = f^{+} + f^{-}$

$$f^{+} := \sum_{n \gg -\infty} c_f^{+}(n) q^n \quad \text{“holomorphic part”}$$

$$f^{-} := \sum_{n<0} c_f^{-}(n, y) q^n \quad \text{“non-holomorphic part”}$$

Harmonic Maass forms

“Definition”

A **mock modular form** *is the holomorphic part of a harmonic Maass form.*

Kac-Wakimoto characters

Case n=1:

Theorem (Bringmann-Ono, '09)

Assuming the notation and hypotheses above, we have that

$$\frac{1}{2}q^{-\frac{s^2}{2(m-1)}+\frac{1}{2}-\frac{m}{24}}tr_s(m;1)-2^{m-2}q^{-\frac{s^2}{2(m-1)}}\frac{\eta^{2m}(2\tau)}{\eta^{2m+1}(\tau)}R(-s\tau;(m-1)\tau)$$

is a non-holomorphic modular function (i.e. weight 0).

Kac-Wakimoto characters

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i.e. up to a q-power, $tr_s(m; 1)$ is a **mixed mock modular form**,

Kac-Wakimoto characters

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i.e. up to a q-power, $tr_s(m;1)$ is a **mixed mock modular form**,

the product of a holomorphic modular and a mock modular form.

Mock theta functions

Universal Mock Theta Functions [Gordon-McIntosh]:

$$g_2(w;q) := \sum_{n\geq 0} \frac{(-q;q)_n q^{\frac{n^2+n}{2}}}{(w;q)_{n+1}(qw^{-1};q)_{n+1}}$$

$$g_3(w;q) := \sum_{n\geq 0} \frac{q^{n^2+n}}{(w;q)_{n+1}(qw^{-1};q)_{n+1}}$$

All original mock-θ's may be obtained from g_2 and g_3.

Mock theta functions

Universal Mock Theta Functions

Example

$$f(-q) = -4qg_3(q;q^4) + \frac{(q^2;q^2)_\infty^7}{(q;q)_\infty^3 (q^4;q^4)_\infty^3}$$

Kac-Wakimoto characters and mock theta functions

Case n=1:

Theorem (F, ‘10)

The following are true.

(i) *For* $m \geq 2$, $s \in \mathbb{Z}$ *we have*

$$g_2\left(q^{\frac{m-1}{4}-\frac{s}{2}}; q^{\frac{m-1}{2}}\right) = \widehat{\Theta}_m(\tau) tr_s(m,1) - \eta_{m,s}(\tau).$$

(ii) *For* $m \in 3\mathbb{N}+1$ *and* $s \in \mathbb{Z}$ *we have*

$$g_3\left(q^{\frac{m-1}{9}-\frac{s}{3}}; q^{\frac{m-1}{3}}\right) = \widehat{\Theta}_m(\tau)\left(q^p \cdot tr_s(m,1) + q^{-p} \cdot tr_{s+\frac{r}{3}}(m,1)\right) - \beta_{m,s}(\tau).$$

(iii) *For* $m \in 6\mathbb{N}+1$ *and* $s \equiv \frac{m-1}{6} \pmod{m-1}$ *we have*

$$g_3\left(q^{\frac{m-1}{9}-\frac{s}{3}}; q^{\frac{m-1}{3}}\right) = \widehat{\Theta}_m(\tau) \cdot q^{\frac{m-1}{36}+\frac{s}{6}} \cdot tr_s(m,1) - \psi_{m,s}(\tau).$$

Kac-Wakimoto characters and mock theta functions

Case n=1:

Corollary. (F, '10)

Ramanujan's mock-θ functions are K-W characters.

Example

$$f(-q) = -4q\widehat{\Theta}_{12}(\tau)\left(q^{\frac{1}{2}}tr_1(13,1)+q^{-\frac{1}{2}}tr_5(13,1)\right) + F(\tau)$$

Kac-Wakimoto characters

Case $n > 1$:

Kac-Wakimoto characters

Case $n > 1$:

- Methods used for $n = 1$ fail.

Kac-Wakimoto characters

Case $n > 1$:

- Methods used for $n = 1$ fail.
- Characters do not appear to be mixed mock modular.

Kac-Wakimoto characters

Case $n > 1$:

Theorem (Bringmann-F, '12)

The Kac-Wakimoto characters $tr_s(m, n)$ *are* **almost mock modular forms** *of weight* 0 *and depth* $n/2$.

Defining almost harmonic Maass forms

Weight κ Maass raising operator:

$$R_\kappa := 2i \frac{d}{d\tau} + \frac{\kappa}{y}$$

$$R_\kappa^n := R_{\kappa+2(n-1)} \circ \cdots \circ R_{\kappa+2} \circ R_\kappa$$

Defining almost harmonic Maass forms

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i.e.

$$\left\{ \begin{matrix} F \text{ weight } \kappa \\ \Delta_\kappa(F) = sF \end{matrix} \right\} \quad \xrightarrow{R_\kappa} \quad \left\{ \begin{matrix} R_\kappa F \text{ weight } \kappa + 2 \\ \Delta_{\kappa+2}(R_\kappa F) = (s+\kappa) R_\kappa F \end{matrix} \right\}$$

Defining almost harmonic Maass forms

Definition (Kaneko-Zagier)

An **almost holomorphic modular form of weight** κ on $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ is a function $F : \mathbb{H} \to \mathbb{C}$ satisfying

1. $F(\gamma\tau) = (c\tau + d)^{\kappa} F(\tau) \ \forall \ \gamma := \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$,
2. $F(\tau) = \displaystyle\sum_{j=0}^{r} \frac{f_j(\tau)}{y^j}$, where $f_j(\tau)$ is holomorphic, $0 \leq j \leq r$.

Defining almost harmonic Maass forms

Example.

$$\widehat{E}_2(\tau) := E_2(\tau) - \frac{3}{\pi y}$$

is an **almost holomorphic modular form** of weight 2.

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Here, the "holomorphic part" is the **quasimodular form**

$$E_2(\tau) := 1 - 24 \sum_{n \geq 1} \sigma_1(n) q^n.$$

$\sigma_1(n) := \sum_{d|n} d$

Almost harmonic Maass forms

Definition (Bringmann-F, '12)

An **almost harmonic Maass form** (almost HMF) of weight $\kappa \in \frac{1}{2}\mathbb{Z}$ and depth r for Γ is a function $f : \mathbb{H} \to \mathbb{C}$ satisfying

1. $f\big|_{\kappa}\gamma(\tau) = \chi(d)f(\tau)$
2. $f = \sum_{j=0}^{r} g_j R_{\kappa+2-\nu}^{j-1}(g)$,

where g = harmonic Maass form weight $\kappa + 2 - \nu$,
g_j = almost holomorphic modular form weight $\nu - 2j$,
$\nu \in \frac{1}{2}\mathbb{Z}$ fixed.

Almost harmonic Maass forms

Structure of almost HMFs:

almost hol. modular ↘

↙ harmonic Maass

$$f = \sum_{j=0}^{r} \underbrace{\underbrace{g_j}_{\text{weight } \nu-2j} \quad \underbrace{R_{\kappa+2-\nu}^{j-1}(g)}_{\text{weight } \kappa-\nu+2j}}_{\text{weight } \kappa}$$

Almost harmonic Maass forms

Examples.

- *Harmonic Mass forms* are almost HMFs of depth $r = 0$.

Almost harmonic Maass forms

Examples.

- *Harmonic Mass forms* are almost HMFs of depth $r = 0$.
- *Almost hol. modular forms* are almost HMFs of depth $r = 1$.

Almost mock modular forms

“Definition”

An **almost mock modular form** *is the holomorphic part of an almost harmonic Maass form.*

Kac-Wakimoto characters (proof sketch)

Sketch of proof of main theorem for $n > 1$*:*

Kac-Wakimoto characters (proof sketch)

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Proposition (Kac, Wakimoto '01)

For $m \geq 2, n \geq 1,$ we have

$$\sum_{s\in\mathbb{Z}} \mathrm{tr}_s(m,n)\zeta^s = \prod_{n=1}^{\infty}(1-q^n)\prod_{k=1}^{\infty}\frac{\left[\left(1+\zeta q^{k-\frac{1}{2}}\right)\left(1+\zeta^{-1}q^{k-\frac{1}{2}}\right)\right]^m}{\left[\left(1-\zeta q^{k-\frac{1}{2}}\right)\left(1-\zeta^{-1}q^{k-\frac{1}{2}}\right)\right]^n}$$

$\zeta = e^{2\pi i z}$

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We re-write the generating function $\sum_{s\in\mathbb{Z}} \mathrm{tr}_s(m,n)\zeta^s$ in terms of

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$$\boxed{\varphi(z;\tau) := \frac{\vartheta\left(z+\frac{1}{2};\tau\right)^m}{\vartheta\left(z;\tau\right)^n}}$$

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where

$$\vartheta(z;\tau) := \sum_{\nu\in\mathbb{Z}+\frac{1}{2}} e^{\pi i\nu}\zeta^\nu q^{\nu^2/2}.$$

Kac-Wakimoto characters (proof sketch)

$\varphi(z;\tau)$ transforms like a **Jacobi form** of index and weight $\frac{m-n}{2}$:

1. $\varphi(z+\lambda\tau+\mu) = q^{-\frac{(m-n)\lambda^2}{2}} e^{-2\pi i(m-n)\lambda z}\varphi(z;\tau), \quad \lambda,\mu\in\mathbb{Z}$
2. $\varphi\left(\frac{z}{c\tau+d};\gamma\tau\right) = \chi(\gamma)(c\tau+d)^{\frac{m-n}{2}} e^{\frac{\pi i c z^2(m-n)}{c\tau+d}}\varphi(z;\tau),$

   $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(2).$

Kac-Wakimoto characters (proof sketch)

Remark.

The Jacobi form $\varphi(z;\tau)$ is **meromorphic**, with poles of order $n > 1$.
Eichler-Zagier theory of **holomorphic** Jacobi forms doesn't directly apply.

Holomorphic Jacobi forms

A theta decomposition:

Proposition (Eichler-Zagier)

Holomorphic Jacobi forms $\phi(z;\tau)$ of weight κ and index M satisfy

$$\phi(z;\tau) = \sum_{b \pmod{2M}} h_b(\tau)\vartheta_{M,b}(z;\tau),$$

where $(h_b(\tau))_{b \pmod{2M}}$ is a vector valued modular form of weight $\kappa - \frac{1}{2}$.

$$\vartheta_{M,b}(z;\tau) := \sum_{\substack{\lambda\in\mathbb{Z} \\ \lambda\equiv b \pmod{2M}}} e^{\frac{\pi i\lambda^2\tau}{2M}+2\pi i\lambda z}$$

Meromorphic Jacobi forms

Question. What is the analogue of the theta decomposition of **holomorphic** Jacobi forms in the **meromorphic** case?

Meromorphic Jacobi forms

1 **Case $\phi(z;\tau)$ has simple or double poles** [Dabholkar-Murthy-Zagier,'12]

$$\phi(z;\tau) = \phi^F(z;\tau) + \phi^P(z;\tau),$$

Meromorphic Jacobi forms

$\boxed{1}$ **Case $\phi(z;\tau)$ has simple or double poles** [Dabholkar-Murthy-Zagier,'12]

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↗ ↖

“finite part” “polar part”

Meromorphic Jacobi forms

1 **Case $\phi(z;\tau)$ has simple or double poles** [Dabholkar-Murthy-Zagier,'12]

The finite part $\phi^F(z;\tau) = \sum_{b \pmod{2M}} h_b(\tau)\vartheta_{M,b}(z;\tau)$

is a **mixed mock Jacobi form**.

Meromorphic Jacobi forms

$\boxed{1}$ **Case $\phi(z;\tau)$ has simple or double poles** [Dabholkar-Murthy-Zagier,'12]

The coefficient functions $h_b(\tau)$ are **mixed mock modular forms.**

Meromorphic Jacobi forms

1 **Case $\phi(z;\tau)$ has simple or double poles** [Dabholkar-Murthy-Zagier,'12]

Remark 1. ϕ^F and ϕ^P are completed to non-holomorphic Jacobi forms $\widehat{\phi^F}$ and $\widehat{\phi^P}$, respectively.

Meromorphic Jacobi forms

$\boxed{1}$ **Case $\phi(z;\tau)$ has simple or double poles** [Dabholkar-Murthy-Zagier,'12]

Remark 1. ϕ^F and ϕ^P are completed to non-holomorphic Jacobi forms $\widehat{\phi^F}$ and $\widehat{\phi^P}$, respectively.

Remark 2. Proof uses Zwegers's mock Jacobi forms of index M.

Meromorphic Jacobi forms

$\boxed{2}$ **Case $\phi(z;\tau)$ has poles of arbitrary order $n \geq 1$.**

Question. What can we say about meromorphic Jacobi forms in general?

Kac-Wakimoto characters (proof sketch)

Theorem (Bringmann-F, '12)

For the meromorphic Jacobi form $\varphi(z;\tau)$ (poles order n), we have the decomposition

$$\varphi(z;\tau) = \varphi^F(z;\tau) + \varphi^P(z;\tau).$$

Kac-Wakimoto characters (proof sketch)

In our setting (Kac-Wakimoto characters),

$$\varphi^F(z;\tau) := \sum_{\ell\in\mathbb{Z}} h_\ell(\tau) q^{\frac{\ell^2}{2(m-n)}} \zeta^\ell = \sum_{\ell \pmod{m-n}} h_\ell(\tau) \vartheta_{\frac{(m-n)}{2},\ell}(z;\tau),$$

$$\varphi^P(z;\tau) := -\sum_{j=1}^{\frac{n}{2}} \frac{\widetilde{D}_{2j}(\tau)}{(2j-1)!} \delta_\varepsilon^{2j-1} \left[f_\varepsilon^{\left(\frac{m-n}{2}\right)}(z;\tau) \right]_{\varepsilon=0}.$$

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quasimodular forms ↘

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↖ mock Jacobi

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quasimodular forms ↘

$$\varphi^P(z;\tau) := -\sum_{j=1}^{\frac{n}{2}} \frac{\widetilde{D}_{2j}(\tau)}{(2j-1)!}\,\delta_\varepsilon^{2j-1}\left[f_\varepsilon^{\left(\frac{m-n}{2}\right)}(z;\tau)\right]_{\varepsilon=0}.$$

diff.operator ↑ ↖ mock Jacobi

Kac-Wakimoto characters (proof sketch)

$$f_{\varepsilon}^{(M)}(z;\tau) := \sum_{k\in\mathbb{Z}} \frac{e^{4\pi iMkz}q^{Mk^2}}{1-e^{2\pi i(z-\varepsilon)}q^k},$$

$$\delta_{\varepsilon} := \frac{1}{2\pi i}\frac{d}{d\varepsilon}.$$

Kac-Wakimoto characters (proof sketch)

Theorem (Bringmann-F, '12)

(1) The functions φ^P *and* φ^F *may be completed to nonholomorphic Jacobi forms* $\widehat{\varphi}^P$ *and* $\widehat{\varphi}^F$*, respectively, of weight and index* $\frac{m-n}{2}$*.*

Kac-Wakimoto characters (proof sketch)

Theorem (Bringmann-F, '12)

(1) The functions φ^P *and* φ^F *may be completed to nonholomorphic Jacobi forms* $\widehat{\varphi}^P$ *and* $\widehat{\varphi}^F$, *respectively, of weight and index* $\frac{m-n}{2}$.

(2) The coefficient functions $h_\ell(\tau)$ *are* **almost mock modular forms** *of weight* $\frac{m-n-1}{2}$ *and depth* $\frac{n}{2}$.

Kac-Wakimoto characters (proof sketch)

Corollary (Bringmann-F, '12)

In particular, the Kac Wakimoto characters $\mathrm{tr}_\ell(m,n)$ *are* **almost mock modular forms** *of weight* 0 *and depth* $\frac{n}{2}$.

Almost harmonic Maass forms

Structure of almost HMFs:

almost hol. modular ↘ ↙ harmonic Maass

$$f = \sum_{j=0}^{r} \underbrace{\underbrace{g_j}_{\text{weight } \nu-2j} \quad \underbrace{R^{j-1}_{\kappa+2-\nu}(g)}_{\text{weight } \kappa-\nu+2j}}_{\text{weight } \kappa}$$

Kac-Wakimoto characters

Example 1. [n=2, double pole]

In this case, the completed

$$\widehat{h}_\ell(\tau)\frac{\eta(\tau)^{m+6}}{\eta(2\tau)^{2m}}$$

is a harmonic Maass form of weight $\frac{3}{2}$ with shadow

$$\vartheta_{\frac{m-2}{2},\ell}(0;\tau).$$

Kac-Wakimoto characters

Example 2. [n=4, pole order 4]

In this case, the completed $\widehat{h}_\ell$ is an almost harmonic Maass form of wgt. $\frac{3}{2}$, depth 2 given by

$$\widehat{h}_\ell(\tau) = \sum_{j=1}^{2} D_{2j}(\tau) R_{\frac{3}{2}}^{j-1}\left(g(\tau)\right),$$

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Example 2. [n=4, pole order 4] (cont.)

where D_4 and D_2 are the *almost holomorphic modular forms*

$$D_4(\tau) = 2^m \frac{\eta(2\tau)^{2m}}{\eta(\tau)^{m+12}},$$

$$D_2(\tau) = 2^m \frac{\eta(2\tau)^{2m}}{\eta(\tau)^{m+12}} \left(\left(-\frac{m}{24} - \frac{1}{6} \right) E_2(\tau) + \frac{m}{6} E_2(2\tau) \right) + \frac{2^{m-3}(4-m)}{\pi y} \frac{\eta(2\tau)^{2m}}{\eta(\tau)^{m+12}}.$$

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Example 2. [n=4, pole order 4] (cont.)

The form $g(\tau)$ is a harmonic Maass form with shadow $\vartheta_{\frac{m-4}{2},\ell}(0;\tau)$.

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Further work:

- Bringmann-Creutzig-Rolen, '14
- Olivetto '14
- Bringmann-F-Mahlburg '15
- Bringmann-Rolen-Zwegers '16
- Brigmann-Olivetto '17

$\vdots$

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Thank you