Generalized Fishburn numbers, torus knots and quantum modularity

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Two goals

- "Generalized Fishburn numbers and torus knots", JCTA **178** (2021), 105355.

- "Quantum modularity of partial theta series with periodic coefficients", Forum Math., to appear.

Fishburn numbers

- The *Fishburn numbers* $\xi(n)$ are the coefficients in the expansion of

$$F(1-q) =: \sum_{n\geq 0} \xi(n) q^n = 1 + q + 2q^2 + 5q^3 + 15q^4 + 53q^5 + \cdots$$

  where $F(q) := \sum_{n\geq 0} (q)_n$ is the Kontsevich-Zagier "strange" series. Here,

$$(a)_n = (a;q)_n := \prod_{k=1}^{n} (1 - aq^{k-1}),$$

  valid for $n \in \mathbb{N}_0 \cup \{\infty\}$.

- $F(q)$ satisfies a "duality" and is a "modular" object.

- $\xi(n)$'s have many nice combinatorial interpretations (see A022493).

Arithmetic properties of $\xi(n)$

- Andrews and Sellers (2016), Guerzhoy, Kent and Rolen (2014), Garvan (2015), Ahlgren and Kim (2015), Straub (2015) studied prime power congruences for $\xi(n)$.

- For example, we have

$$\xi(5^r m - 1) \equiv \xi(5^r m - 2) \equiv 0 \pmod{5^r},$$

$$\xi(7^r m - 1) \equiv 0 \pmod{7^r}$$

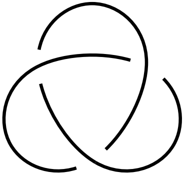
and

$$\xi(11^r m - 1) \equiv \xi(11^r m - 2) \equiv \xi(11^r m - 3) \equiv 0 \pmod{11^r}.$$

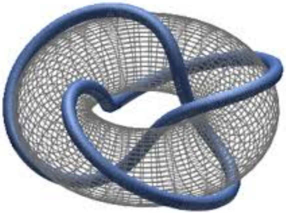
- Our first goal is to generalize $\xi(n)$ using knot theory.

Knots

- A *knot* K is an embedding of a circle in $\mathbb{R}^3$. For example, the right-handed trefoil knot is given by



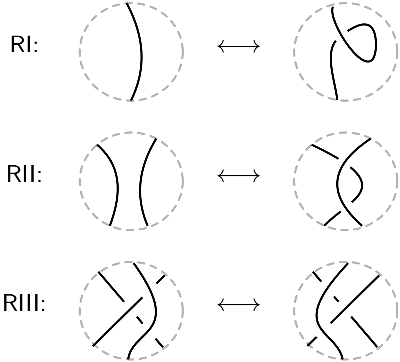
- We will consider the family of *torus knots* $T(p,q)$:



Knots

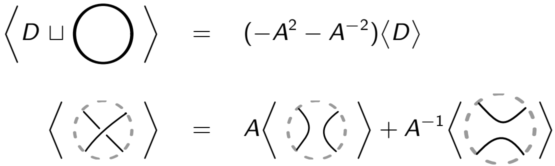
Theorem (Reidemeister, 1927)

Let K and K' be two knots with diagrams D and D'. Then K is isotopic to K' in $\mathbb{R}^3$ if and only if D is related to D' by a sequence of isotopies of $\mathbb{R}^2$ and the moves RI, RII and RIII given by the following:



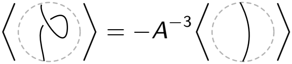
The Jones polynomial

- The *Kauffman bracket* $\langle D \rangle$ of D is defined by



$$\Big\langle \text{empty diagram} \Big\rangle = 1.$$

- $\langle D \rangle$ is invariant under RII and RIII, but not RI as

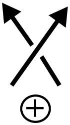


The Jones polynomial

- The Jones polynomial $V(K) = V(K;q)$ is given by

$$V(K) = \frac{1}{(-A^2 - A^{-2})}(-A)^{-3w(D)}\langle D\rangle\Big|_{A^2=q^{-1/2}}$$

where

$w(D) = \quad \#$  $\quad - \quad \#$ 

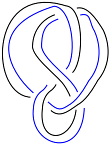
is the "writhe" of D.

- $V(K)$ is invariant under RI, RII and RIII.

The colored Jones polynomial

- The colored Jones polynomial $J_N(K;q)$ is a linear combination of cablings of D using Chebyshev polynomials: $S_1(x) = 1$, $S_2(x) = x$, $S_N(x) = xS_{N-1}(x) - S_{N-2}(x)$.

- For example, $S_3(x) = x^2 - 1$. So, we have

$$J_3(4_1;q) \quad = \quad \star \left\langle \vphantom{\int} \right. $$  $$\left. \right\rangle - 1$$

- The $N = 2$ case recovers the Jones polynomial.

Expansions

- (Habiro, 2008) For *any* knot K, we have the "cyclotomic expansion"

$$J_N(K;q) = \sum_{n\geq 0} \underbrace{C_n(K;q)}_{\in\mathbb{Z}[q^{\pm 1}]} (q^{1+N})_n (q^{1-N})_n.$$

- (Masbaum, 2003) For example,

$$J_N(\text{trefoil}^*;q) = \sum_{n\geq 0} q^n (q^{1+N})_n (q^{1-N})_n.$$

- (Habiro (2000), T. Lê (2003)) The "non-cyclotomic" expansion is

$$J_N(\text{trefoil};q) = q^{1-N} \sum_{n\geq 0} q^{-nN} (q^{1-N})_n.$$

Expansions

- (Bryson, Ono, Pitman, Rhoades, 2012, PNAS) We have the "duality"

$$F(\zeta_N^{-1}) = U(-1; \zeta_N)$$

where

$$U(x; q) = \sum_{n \geq 0} (-xq)_n (-x^{-1}q)_n q^{n+1}.$$

- For *any* knot K, we have $J_N(K; q^{-1}) = J_N(K^*; q)$. Thus,

$$F(\zeta_N^{-1}) \underbrace{=}_{\text{Habiro, Lê}} J_N(\text{trefoil}; \zeta_N^{-1})\zeta_N = J_N(\text{trefoil}^*; \zeta_N)\zeta_N \underbrace{=}_{\text{Masbaum}} U(-1; \zeta_N).$$

- (Hikami, Lovejoy, 2015) $U(x; q)$ is a (mixed) mock modular form (when x is a root of unity $\neq -1, \pm i$) as

$$(1-x)U(-x; q) = \frac{q}{(q)_\infty} \Big(\sum_{r,n \geq 0} - \sum_{r,n<0} \Big) (-1)^{n+r} x^{-r} q^{\frac{n(3n+5)}{2} + 2nr + \frac{r(r+3)}{2}}.$$

Our situation

- Consider the family of torus knots $T(3, 2^t)$, $t \geq 1$. In 2016, Konan proved

$$J_N(T(3,2^t);q) = (-1)^{h''(t)} q^{2^t-1-h'(t)-N} \sum_{n\geq 0} (q^{1-N})_n q^{-Nnm(t)}$$
$$\times \sum_{3\sum_{\ell=1}^{m(t)-1} j_\ell \ell \equiv 1 \pmod{m(t)}} (-q^{-N})^{\sum_{\ell=1}^{m(t)-1} j_\ell} q^{\frac{-a(t)+\sum_{\ell=1}^{m(t)-1} j_\ell \ell}{m(t)} + \sum_{\ell=1}^{m(t)-1} \binom{j_\ell}{2}}$$
$$\times \sum_{k=0}^{m(t)-1} q^{-kN} \prod_{\ell=1}^{m(t)-1} \underbrace{\begin{bmatrix} n + I(\ell \leq k) \\ j_\ell \end{bmatrix}}_{q\text{-binomial coefficient}}.$$

- Let $\mathcal{F}_t(q) := (-1)^{h''(t)} q^{-h'(t)} \sum_{n\geq 0} (q)_n \sum_{j_\ell}{}' q^v \prod_{\ell=1}^{m(t)-1} \begin{bmatrix} n + I(\ell \leq k) \\ j_\ell \end{bmatrix}$.

Our situation

- We have $\mathcal{F}_1(q) = F(q)$ and

$$\zeta_N^{2^t-1}\mathcal{F}_t(\zeta_N) = J_N(T(3,2^t);\zeta_N).$$

- Write

$$\mathcal{F}_t(1-q) =: \sum_{n\geq 0} \xi_t(n)q^n.$$

- For example,

$$\mathcal{F}_2(1-q) = 1 + 3q + 11q^2 + 50q^3 + 280q^4 + 1890q^5 + \cdots$$

and

$$\mathcal{F}_3(1-q) = 1 + 7q + 49q^2 + 420q^3 + 4515q^4 + 59367q^5 + \cdots .$$

First result

- Let

$$\chi_t(n) := \begin{cases} 1 & \text{if } n \equiv 2^{t+1}-3,\ 3+2^{t+2} \pmod{3 \cdot 2^{t+1}}, \\ -1 & \text{if } n \equiv 2^{t+1}+3,\ 2^{t+2}-3 \pmod{3 \cdot 2^{t+1}}, \\ 0 & \text{otherwise} \end{cases}$$

  and for $s \in \mathbb{N}$, define

$$S_{t,\chi_t}(s) = \left\{ 0 \leq j \leq s-1 : j \equiv \frac{n^2 - (2^{t+1}-3)^2}{3 \cdot 2^{t+2}} \pmod{s} \text{ where } \chi_t(n) \neq 0 \right\}.$$

Theorem (Bijaoui, Boden, Myers, –, Rushworth, Tronsgard, Zhou)

If $p \geq 5$ is a prime and $j \in \{1, 2, \dots, p-1-\max S_{t,\chi_t}(p)\}$, then

$$\xi_t(p^r m - j) \equiv 0 \pmod{p^r}$$

for all natural numbers r, m and $t \geq 1$.

Sketch of proof

- Prove a new "strange identity". Recall that (Zagier, 2001)

$$F(q) \text{``}=\text{''} -\frac{1}{2}\sum_{n\geq 1} n \underbrace{\left(\frac{12}{n}\right)}_{\chi_1(n)} q^{\frac{n^2-1}{24}}.$$

- We first prove that

$$\mathcal{F}_t(q) \text{``}=\text{''} -\frac{1}{2}\sum_{n\geq 0} n\chi_t(n) q^{\frac{n^2-(2^{t+1}-3)^2}{3\cdot 2^{t+2}}}.$$

- This follows from the following key identity ...

Key identity

$$\frac{1}{2}\sum_{n\geq 0} n\chi_t(n) q^{\frac{n^2-(2^{t+1}-3)^2}{3\cdot 2^{t+2}}} - \frac{2^{t+1}-3}{2}(q^{2^t-1}, q^{2^t+1}, q^{2^{t+1}}; q^{2^{t+1}})_\infty (q^2, q^{2^{t+2}-2}; q^{2^{t+2}})_\infty$$

$$= (-1)^{h''(t)+1} q^{-h'(t)} \sum_{n\geq 0}\Big[(q)_n - (q)_\infty\Big]$$

$$\times \sum_{j_\ell}{}' (-1)^{\sum_{\ell=1}^{m(t)-1} j_\ell} q^v \sum_{k=0}^{m(t)-1} \prod_{\ell=1}^{m(t)-1} \begin{bmatrix} n + I(\ell \leq k) \\ j_\ell \end{bmatrix}$$

$$+ (-1)^{h''(t)+1} q^{-h'(t)} (q)_\infty \left(\sum_{i=1}^{\infty} \frac{q^i}{1-q^i}\right) \sum_{n\geq 0} b_{n,t}(q)$$

$$+ (-1)^{h''(t)} q^{-h'(t)} (q)_\infty \sum_{n\geq 0} (n - h(t)) b_{n,t}(q)$$

where $b_{n,t}(q)$ is an explicit q-multisum.

Sketch of proof

- Let p be a prime ≥ 5 and $n \geq r$ be an integer. Consider the truncation of $\mathcal{F}_t(1-q)$, then its p-dissection:

$$\begin{aligned}\mathcal{F}_t(1-q; pn-1) &= \sum_{i=0}^{p-1} (1-q)^i A_{t,p}(pn-1, i, (1-q)^p) \\ &= \sum_{i \in S_{t,\chi_t}(p)} (1-q)^i A_{t,p}(pn-1, i, (1-q)^p) \\ &\quad + \sum_{i \notin S_{t,\chi_t}(p)} (1-q)^i A_{t,p}(pn-1, i, (1-q)^p) \\ &=: \sum\nolimits_1 + \sum\nolimits_2 .\end{aligned}$$

- The coefficient of $q^{p^r m - j}$ in the summand of $\sum_1$ is $\equiv 0 \pmod{p^r}$.

- (AKL, 2019) Strange identity implies $\sum_2 \equiv O(q^{pn-(p-1)(r-1)}) \pmod{p^r}$.

Quantum modularity

Definition (Zagier, 2010)

A quantum modular form of weight $k \in \frac{1}{2}\mathbb{Z}$ is a function $g : \mathbb{Q} \to \mathbb{C}$ such that for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$,

$$r_\gamma(\alpha) := g(\alpha) - (c\alpha + d)^{-k} g\left(\frac{a\alpha+b}{c\alpha+d}\right)$$

has "nice" properties (e.g., continuity or analyticity).

- (Zagier, 2010) The function $\phi(\alpha) := e^{\frac{\pi i \alpha}{12}} F(e^{2\pi i \alpha})$ is a quantum modular form of weight $3/2$ with respect to $SL_2(\mathbb{Z})$.
- Suitable modifications can be made to restrict the domain of r_γ to appropriate subsets of $\mathbb{Q}$ and allow both multiplier systems and transformations on subgroups of $SL_2(\mathbb{Z})$.

Second result

- For $N \in \mathbb{N}$, let

$$\Gamma_1(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : c \equiv 0 \pmod{N},\ a \equiv d \equiv 1 \pmod{N} \right\}$$

and set $s_t := \frac{(2^{t+1}-3)^2}{3 \cdot 2^{t+2}}$.

Theorem (Goswami, –)

For an integer $t \geq 2$ and $\alpha \in \mathbb{Q}$, $\phi_t(\alpha) := e^{2\pi i s_t \alpha} \mathfrak{F}_t(e^{2\pi i \alpha})$ is a quantum modular form of weight $3/2$ on $A_{3.2^{t+1}} := \{\alpha \in \mathbb{Q} : \alpha \text{ is } \Gamma_1(3 \cdot 2^{t+2})\text{-equivalent to } 0 \text{ or } i\infty\}$ with respect to $\Gamma_1(3 \cdot 2^{t+2})$.

Idea of proof

- Let $\theta_t(z) := \sum_{n\geq 0} \chi_t(n) q^{\frac{n^2}{3\cdot 2^{t+2}}}$ and $\Theta_t(z) := \sum_{n\geq 0} n\chi_t(n) q^{\frac{n^2}{3\cdot 2^{t+1}}}$.
- For $\alpha \in A_{3.2^{t+1}}$, we show that

$$\Theta_t(\alpha) - (\Theta_t|_{\frac{3}{2},\chi_t}\gamma)(\alpha) =: r_{\gamma,t}(\alpha)$$

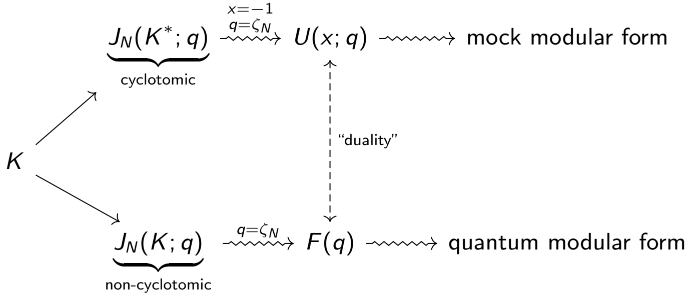
where

$$r_{\gamma,t}(z) = -\frac{\sqrt{3\cdot 2^{t+1}}\cdot e^{\frac{i\pi}{4}}}{2\pi}\int_{\gamma^{-1}(i\infty)}^{i\infty} \theta_t(\tau)(\tau - \bar{z})^{-\frac{3}{2}}\, d\tau.$$

- Apply the "strange identity" for $\mathcal{F}_t(q)$.

Future work

- Consider the picture:



- $T(3,2)$: Zagier $\rightsquigarrow F\checkmark$, HL $\rightsquigarrow U\checkmark$
  $T(2,2t+1)$: Hikami $\rightsquigarrow F\checkmark$, HL $\rightsquigarrow U$?
  $T(3,2^t)$: Goswami, – $\rightsquigarrow F\checkmark$, NO U *yet*!!
  Satellite knots? Hyperbolic knots?

Future work

- We have

$$(q)_\infty(-1)^{h''(t)}q^{-h'(t)}\sum_{j_\ell}{}'(-1)^{\sum_{\ell=1}^{m(t)-1}j_\ell}\frac{q^{v}}{(q)_{j_1}\cdots(q)_{j_{m(t)-1}}}$$
$$=(q^{2^t-1},q^{2^t+1},q^{2^{t+1}};q^{2^{t+1}})_\infty(q^2,q^{2^{t+2}-2};q^{2^{t+2}})_\infty.$$

  This recovers an identity of Slater:

$$(q)_\infty\sum_{n\geq 0}\frac{q^{2n(n+1)}}{(q)_{2n+1}}=(q^3,q^5,q^8;q^8)_\infty(q^2,q^{14};q^{16})_\infty.$$

  Proof using Bailey pairs? Combinatorial proof?

- The numbers $\xi_t(n)$ appear to be positive. What are they counting?

- Thank you!