Counting Quaternion Algebras, with applications to Spectral Geometry

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Overview

Our Goal: Prove effective versions of "rigidity" results for arithmetic, hyperbolic 2- & 3-manifolds

Key Idea:

$$\left\{\begin{array}{l}\text{Maximal subfields}\\ \text{of quaternion algebras}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\text{lengths of}\\ \text{geodesics on}\\ \text{arithmetic hyperbolic}\\ \text{2- \& 3-manifolds}\end{array}\right\}$$

number theory

geometry

Today's plan:

① Introduce quaternion algs.

② Our results on counting quaternion algs.

③ Geometric Background

④ How to construct surfaces from quaternion algebras

⑤ Our results on surfaces

⑥ Connections to bounded gaps between primes

I. Quaternion algebras & orders

Theorem (Hamilton, 1843)

The $\mathbb{R}$*-algebra* $\mathbb{H}$ *with basis* $\{1, i, j, ij\}$ *and defining relations*

$$i^2 = -1 \qquad j^2 = -1 \qquad ij = -ji$$

is a four-dimensional division algebra.

Brougham Bridge:

<u>Notation</u>: write $(-1, -1, \mathbb{R})$ instead of $\mathbb{H}$.

Can replace w/ other units in $\mathbb{R}$

Can be replaced w/ other fields of char 0

<u>Ex</u> $(\underline{1}, \underline{1}, \mathbb{R})$ has $i^2 = j^2 = 1$ &

$$ij = -ji.$$

Observe:

$$(1, 1, \mathbb{R}) \cong M_2(\mathbb{R})$$

$$i \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$j \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

In general:

$$(a,b,\mathbb{R}) \cong \begin{cases} \mathbb{H} & \text{if } a,b<0 \\ M_2(\mathbb{R}) & \text{otw} \end{cases}$$

Thus, $(a,b,\mathbb{R})$ is either a division algebra or isomorphic to $M_2(\mathbb{R})$.

* If we replace $\mathbb{R}$ w/ other fields:

Theorem (Wedderburn)

For any field F, if the F-algebra (a,b,F) is not a division algebra then $(a,b,F) \cong \mathrm{M}_2(F)$.

Extension of Scalars

Let K be a field, K'/K a field ext.

If $B = (a, b, K)$ is a quaternion alg $/K$, then

$$B \otimes_K K' = (a, b, K')$$

is a quaternion alg $/K'$.

Def If B is a quaternion alg $/K$, let

$$B_{\mathfrak{p}} := B \otimes_K K_{\mathfrak{p}}.$$

we say B is ramified at $\mathfrak{p}$ if $B_{\mathfrak{p}}$ is the unique division alg $/K_{\mathfrak{p}}$. Otw, B splits in $\mathfrak{p}$.

Def Let $\mathrm{Ram}(B)$ denote the (finite) set of primes at which B is ramified. The discriminant of B is the ideal defined by

$$\Delta(B) := \prod_{\mathfrak{p} \in \mathrm{Ram}(B)} \mathfrak{p}.$$

Def Let B be a quaternion alg $/K$. The reduced norm of B is the composite map

$$B \hookrightarrow M_2(\mathbb{C}) \xrightarrow{\det} \mathbb{C}$$

Let K be a number field w/ ring of integers $\mathcal{O}_K$.

Def An order of a K-alg. is a subring which is also a f.g. $\mathcal{O}_K$-module containing an K-basis of the algebra.

Def An order is maximal if it is not properly contained in any other order.

Ex $M_2(\mathbb{Z})$ is a max. order of $M_2(\mathbb{Q})$.

II. Counting Quaternion Algebras w/ Prescribed Embeddings



Theorem (Linowitz, McReynolds, Pollack, T., 2018)

Fix a number field k, and fix quadratic extensions $L_1, L_2, \ldots, L_r$ of k. Let L be the compositum of the L_i, and suppose that $[L : k] = 2^r$. The number of quaternion algebras over k with discriminant having norm less than x and which admit embeddings of all of the L_i is

$$\sim \delta \cdot x^{1/2}/(\log x)^{1-\frac{1}{2^r}},$$

as $x \to \infty$. Here δ is a positive constant depending only on the L_i and k.

Proof Sketch:

① Construct a Dirichlet Series whose Coefficients Count the quaternion algebras w/ the properties we are interested in

② Apply Delange's Tauberian Thm

Theorem (Delange's Tauberian Theorem)

Let $G(s) = \sum \frac{a_N}{N^s}$ be a Dirichlet series satisfying:

1. $a_N \geq 0$ *for all* N *and* $G(s)$ *converges for* $\Re e(s) > \rho$.
2. $G(s)$ *can be continued to an analytic function in the closed half-plane* $\Re e(s) \geq \rho$ *except possibly for a singularity at* $s = \rho$.
3. *There is an open neighborhood of* ρ *and functions* $A(s), B(s)$ *analytic at* $s = \rho$ *with* $G(s) = A(s)/(s-\rho)^\beta + B(s)$ *at every point in this neighborhood having* $\Re e(s) > \rho$.

Then as $x \to \infty$ *we have*

$$\sum_{N \leq x} a_N = \left(\frac{A(\rho)}{\rho \Gamma(\beta)} + o(1) \right) x^\rho \log(x)^{\beta - 1}.$$

③ Obtain an asymptotic

Handling the embeddings of the L_i's

From Class Field Theory:

If B/k is a quaternion algebra and L/k is a quadratic extension, then L embeds into B iff no prime of k that divides the discriminant of B splits in L/k.

$\leftarrow$ Can use a deep result of M. Matchett Wood to model the splitting of finitely many primes as mutually independent events over the class of random quadratic extensions of k.

III. Geometric Motivation

Def A closed geodesic on a Riemannian manifold M is a map

$$c : S^1 \to M$$

that locally yields the shortest distance between two points.

Examples

Let M be a compact Riemannian manifold.

The Laplace eigenvalue spectrum of M, denoted $\mathcal{E}(M)$, is the multiset of eigenvalues of the Laplacian of M.

The geodesic length spectrum of M, denoted $LS(M)$, is the multiset of lengths of closed geodesics of M.

Let M, N be Compact Riemannian manifolds.

3 Notions of "equivalence":

① M & N are Commensurable if they have a common finite degree covering space.

② M & N are isometric if $\exists$ an isometry between them.

③ M & N are isospectral if $\mathcal{E}(M) = \mathcal{E}(N)$ & length isospectral $LS(M) = LS(N)$

"Inverse" questions

To what extent do the spectra of M determine its geometry/topology?

Examples:

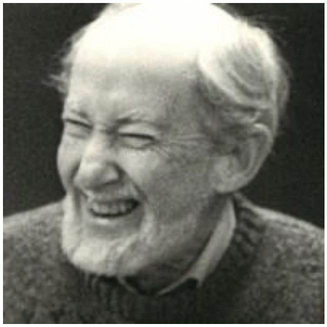
① If $LS(M) = LS(N)$, are M & N isometric? NO

② If $LS(M) = LS(N)$, are M & N commensurable?

↳ Arithmetic: Yes

↳ Non-arithmetic: ???

Can you hear the shape of a drum?



Leon Green (1960) asked if the spectrum of M determines its isometry class.

The spectrum of M is essentially the collection of frequencies produced by a drumhead shaped like M.

Mark Kac (1966) popularized this question for planar domains:

Can you hear the shape of a drum?

Answer: No!

Theorem (Gordon, Webb, Wolpert, 1992)

One cannot hear the shape of a drum.

i.e., isospectral $\not\Rightarrow$ isometric

Hyperbolic Surfaces

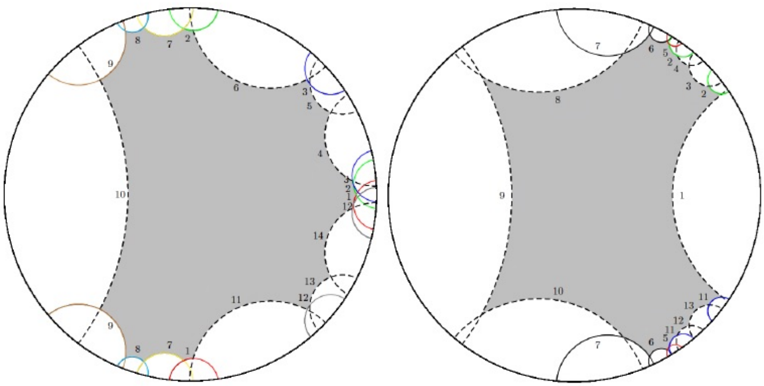
The hyperbolic plane $\to \mathbb{H}^2$ is a simply connected surface w/ constant curvature -1 and can be modeled by the unit disc:

M.C. Escher's Circle Limit IV

Results for Hyperbolic Surfaces

Theorem (Vigneras, 1980)

There exist isospectral non-isometric hyperbolic 2- and 3-manifolds.



A pair of isospectral but non-isometric hyperbolic 2-orbifolds (due to B. Linowitz and J. Voight).

IV. Constructing Arithmetic Manifolds

Elementary results from geometric group theory:

- $Isom^{+}(\mathbf{H}^2) \cong PSL_2(\mathbb{R})$.
- Every orientable hyperbolic 2-manifold is of the form $\mathbf{H}^2/\Gamma$ for some discrete subgroup Γ of $PSL_2(\mathbb{R})$.

We want to generalize the following construction of $PSL_2(\mathbb{Z})$:

$$\overset{B}{M_2(\mathbb{Q})} \supset M_2(\mathbb{Z}) \rightarrow SL_2(\mathbb{Z}) \rightarrow PSL_2(\mathbb{Z})$$

Replace w/ quat. alg. B/K in which a unique real prime splits

replace w/ maximal quat. order $\mathcal{O}$ of B

Consider the embedding $\rho : B \rightarrow M_2(\mathbb{R})$.

Restricting ρ to $\mathcal{O}^1$
& Projecting onto $PSL_2(\mathbb{R})$
gives an embedding:

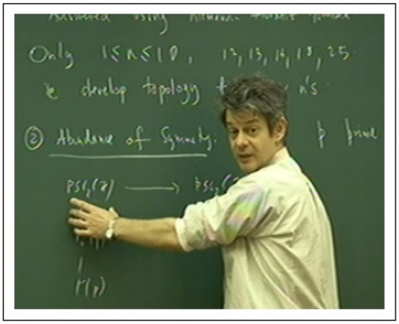
$$\bar{\rho} : \mathcal{O}^1 \to PSL_2(\mathbb{R}).$$

Notice:

* $\bar{\rho}(\mathcal{O}^1)$ is a discrete s.g. of isometries w/ finite covolume
* If B is a division alg, then $\bar{\rho}(\mathcal{O}^1)$ is cocompact.
* If $\bar{\rho}(\mathcal{O}^1)$ is torsion-free then $\mathbb{H}^2/\bar{\rho}(\mathcal{O}^1)$ is a hyperbolic 2-manifold.

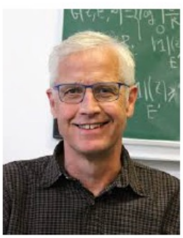
hyp. surfaces commensurable w/ this are arithmetic.

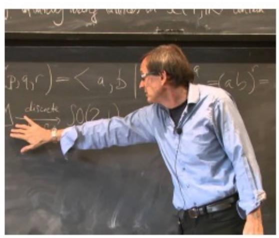
If $LS(M) = LS(N)$, are M & N Commensurable?



Theorem (Reid, 1992)

If M is an arithmetic, hyperbolic surface and $\mathrm{LS}(M) = \mathrm{LS}(N)$ then M and N are commensurable.





Theorem (Chinburg, Hamilton, Long, Reid, 2008)

If M and N are arithmetic hyperbolic 3-manifolds and $\mathrm{LS}(M) = \mathrm{LS}(N)$ then M and N are commensurable.

on the other hand...

Theorem (Futer and Millichap, 2016)

For every sufficiently large $n > 0$ there exists a pair of non-isometric finite-volume hyperbolic 3-manifolds $\{M, N\}$ such that:

1. $\mathrm{vol}(M) = \mathrm{vol}(N)$.
2. *The (complex) length spectra of M and N agree up to length n.*
3. *M and N have at least e^n/n closed geodesics up to length n.*
4. *M and N are not commensurable.*

Summary

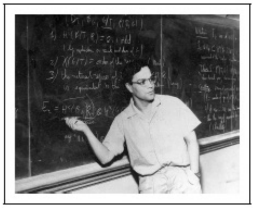
In summary:

- When two **arithmetic** hyperbolic 2- or 3-manifolds have the same geodesic lengths, they are commensurable.
- Two **non-arithmetic** hyperbolic 2- or 3- manifolds can have a great deal in common (same volume, lots of overlap in geodesic lengths) but still not be commensurable.

Motivating Questions:

- Can we make Reid's result effective?
- If the length spectra have a great deal of overlap, must the corresponding arithmetic, hyperbolic 2-manifolds be commensurable?

V. Our Results On Surfaces



Theorem (Borel's finiteness result)

For each $V \in \mathbb{R}_{\geq 0}$ there are only finitely many arithmetic hyperbolic 2- manifolds of volume at most V.

↳ Consequence: $\exists\, L(V) \in \mathbb{R}_{\geq 0}$ such that if M, N are arithmetic hyperbolic surfaces of area $\leq V$ and have the same geodesic lengths up to $L(V)$ then M & N are Commensurable.

Making Reid's "Isospectral ⇒ Commensurable" Result Effective

Theorem (Linowitz, McReynolds, Pollack, T.)

If M and N are **arithmetic hyperbolic** *surfaces of area at most V then*

$$L(V) \leq c_1 e^{c_2 \log(V) V^{130}}$$

for absolute, effectively computable constants c_1 and c_2.

Translating Our Number Theoretic Result to a Geometric Result

Theorem (Linowitz, McReynolds, Pollack, T., 2018)

Fix a number field k, and fix quadratic extensions $L_1, L_2, \ldots, L_r$ of k. Let L be the compositum of the L_i, and suppose that $[L : k] = 2^r$. The number of quaternion algebras over k with discriminant having norm less than x and which admit embeddings of all of the L_i is

$$\sim \delta \cdot x^{1/2}/(\log x)^{1-\frac{1}{2^r}},$$

as $x \to \infty$. Here δ is a positive constant depending only on the L_i and k.

Let $\pi(V, S)$ denote the maximum cardinality of a collection of pairwise non-commensurable arithmetic hyperbolic 2–orbifolds **derived from quaternion algebras**, each of which has volume less than V and geodesic length spectrum containing S. finite set

Theorem (Linowitz, McReynolds, Pollack, T., 2018)

If $\pi(V, S) \to \infty$ as $V \to \infty$, then there are integers $1 \leq r, s \leq |S|$ and constants $c_1, c_2 > 0$ such that

$\{\ell_1, \ldots, \ell_r\}$

$$\frac{c_1 V}{\log(V)^{1-\frac{1}{2^r}}} \leq \pi(V, S) \leq \frac{c_2 V}{\log(V)^{1-\frac{1}{2^s}}}$$

for all sufficiently large V.

Pf Sketch

Let M be an arith., hyp. 2-manifold arising from (K, B) w/ fundamental group $\Gamma < PSL_2(\mathbb{R})$. There is a bijection:

$$\{C_\gamma : S^1 \to M\} \longleftrightarrow \{[\gamma]_\Gamma : \gamma \in \Gamma\}$$

closed geodesics on M

Γ conjugacy classes of $\gamma \in \Gamma$

Geodesic lengths $\ell(C_\gamma)$ are given by

$$\cosh \frac{\ell(C_\gamma)}{2} = \pm \frac{Tr(\gamma)}{2}$$

Let $\lambda_\gamma :=$ Unique eigenvalue of γ w/ $|\lambda_\gamma| > 1$.

* Each closed geodesic C_γ determines a maximal subfield K_γ of the quat. Alg. B:

$$K_\gamma = K(\lambda_\gamma)$$

$$\{\ell_1, \dots, \ell_r\} \rightleftarrows \{L_1, \dots, L_r\}$$

Let $\pi(V,S)$ denote the maximum cardinality of a collection of pairwise non-commensurable arithmetic hyperbolic 2–orbifolds **derived from quaternion algebras**, each of which has volume less than V and geodesic length spectrum containing S.

Theorem (Linowitz, McReynolds, Pollack, T., 2018)

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$$\frac{c_1 V}{\log(V)^{1-\frac{1}{2^r}}} \leq \pi(V,S) \leq \frac{c_2 V}{\log(V)^{1-\frac{1}{2^s}}}$$

for all sufficiently large V.

Conclusions:

① There are lots of pairwise non-commensurable 3-manifolds w/ a great deal of overlap in their geodesic lengths.

② The counting function looks a bit like the count of prime numbers...

Bounded Gaps Between Volumes of Manifolds

Let $\pi(V, S)$ denote the maximum cardinality of a collection of pairwise non-commensurable arithmetic hyperbolic 2–orbifolds derived from quaternion algebras, each of which has volume less than V and geodesic length spectrum containing S.

Theorem (Linowitz, McReynolds, Pollack, T., 2017)

Suppose that $\pi(V, S) \to \infty$ as $V \to \infty$. Then, for every $k \geq 2$, there is a constant $C > 0$ such that there are infinitely many k–tuples $M_1, \ldots, M_k$ of arithmetic hyperbolic 2–orbifolds which are pairwise non-commensurable, have length spectra containing S, and volumes satisfying $|\mathrm{vol}(M_i) - \mathrm{vol}(M_j)| < C$ for all $1 \leq i, j \leq k$.

Pf Sketch

Borel's Covolume Formula:

$$\mathrm{vol}(\mathbf{H}^2/\Gamma^1_{\mathcal{O}}) = \frac{|\Delta_K|^{3/2}\zeta_K(2)}{(4\pi^2)^{n_K-1}} \prod_{P \in \mathrm{Ram}_f(B)} (N(P)-1)\,.$$

Only depends on K

Only depends on B

BCF ⇒ If two orbifolds have the same field of def K, but their associated B's ramify at different primes, their volumes will only differ by some function of the N(p)'s.

Moral: primes w/ gaps between them produce orbifolds w/ volumes that have bdd gaps between them!

What's missing?

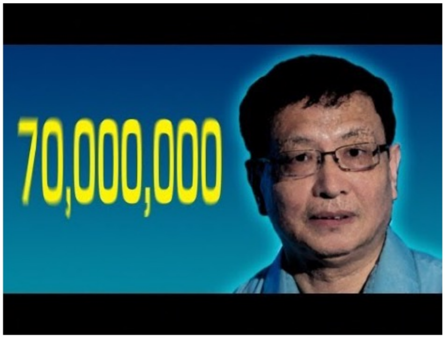
* Need orbifolds to have length spectra containing δ.

$\Updownarrow$

quadratic extensions k_δ embed into the B's.

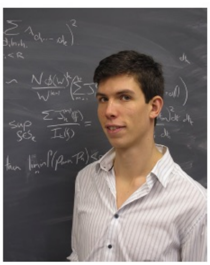
need bounded gaps between primes in Chebotarev sets

Bounded gaps between primes



Corollary (Zhang, 2013)

There are infinitely many pairs of primes that differ by at most $70,000,000$.



Theorem (Maynard-Tao, November 2013)

Let $m \geq 2$. *There for any admissible* k*-tuple* $\mathcal{H} = (h_1, ..., h_k)$ *with "large enough"* k *(relative to* m*), there are infinitely many* n *such that at least* m *of* $n + h_1, ..., n + h_k$ *are prime.*

Tuples of primes

Q: Are the ∞ly many tuples of primes $(p+h_1, \ldots, p+h_k)$?

* Some tuples clearly fail:

Ex $p, p+2, p+4$

Which k-tuples might work?

Definition

We say that a k-tuple $(h_1, \ldots, h_k)$ of nonnegative integers is *admissible* if it doesn't cover all of the possible remainders $\pmod{p}$ for any prime p.

Ex $(0, 2, 6, 8, 12)$

Residue classes not covered:

$1 \pmod{2}$

$1 \pmod{3}$

$4 \pmod{5}$

$3 \pmod{7}$

$3 \pmod{11}$

Bounded gaps between primes in Chebotarev Sets

Theorem (Thorner, 2014)

Let $K/\mathbb{Q}$ be a Galois extension of number fields with Galois group G and discriminant Δ, and let $\mathcal{C}$ be a conjugacy class of G. Let $\mathcal{P}$ be the set of primes $p \nmid \Delta$ for which $\left(\frac{K/\mathbb{Q}}{p}\right) = \mathcal{C}$. Then there are infinitely many pairs of distinct primes $p_1, p_2 \in \mathcal{P}$ such that $|p_1 - p_2| \le c$, where c is a constant depending on $G, \mathcal{C}, \Delta$.

Chebotarev Sets

Some examples of Chebotarev sets:

- The set of primes $p \equiv 1 \pmod{3}$ for which 2 is a cube $\pmod{p}$.
- Fix $n \in \mathbb{Z}^+$. The set of primes expressible in the form $x^2 + ny^2$.
- Let τ be the Ramanujan tau function. The set of primes p for which $\tau(p) \equiv 0 \pmod{d}$ for any positive integer d.
- The set of primes p for which $\#E(\mathbb{F}_p) \equiv p + 1 \pmod{d}$ for any positive integer d.

Generalizing Thorner's work

Theorem (Linowitz, McReynolds, Pollack, T., 2017)

Let L/K be a Galois extension of number fields, let $\mathcal{C}$ be a conjugacy class of $\mathrm{Gal}(L/K)$, and let k be a positive integer. Then, for a certain constant $c = c_{L/K,\mathcal{C},k}$, there are infinitely many k–tuples $P_1, \ldots, P_k$ of prime ideals of K for which the following hold:

1. $\left(\frac{L/K}{P_1}\right) = \cdots = \left(\frac{L/K}{P_k}\right) = \mathcal{C}$,
2. $P_1, \ldots, P_k$ *lie above distinct rational primes,*
3. *each of* $P_1, \ldots, P_k$ *has degree* 1,
4. $|N(P_i) - N(P_j)| \leq c$, *for each pair of* $i, j \in \{1, 2, \ldots, k\}$.

Thorner's theorem is the case where $K = \mathbb{Q}$.

Summary

In summary:

- There are lots of pairwise non-commensurable arithmetic hyperbolic 2-manifolds with a great deal of overlap in their geodesic lengths.
- One might wonder if the volumes of these manifolds are getting further and further apart. We show that they are not.
- Our proof involves generalizing the work of Maynard-Tao and Thorner, and then translating it to a geometric setting.

Thank

You!