Lecture 8: 10 November 2020

- modular forms
- real analytic modular forms
- harmonic Maass forms
  and mock modular forms
- examples, Python code
- proving identities

Lecture 8: 10 November 2020

- modular forms
  - definition
  - Eisenstein series
  - proving identities
  - The Eisenstein series $E_2(\tau)$
- mock theta functions
  - Ramanujan's vague definition
  - Real analytic modular forms
- harmonic Maass forms
  - definitions and terminology
  - Eisenstein series E_2
  - Real analytic modular forms (Zwegers)
  - Hurwitz class numbers (Zagier)
    - Python code!
- Proving mock theta conjectures

Notation $q \in \mathbb{C}, \quad 0<|q|<1$

$$(x)_\infty = (x;q)_\infty = \prod_{i=0}^{\infty} (1-q^i x)$$

$$j(z;q) = \sum_{n=-\infty}^{\infty} (-1)^n z^n q^{\binom{n}{2}}$$

$$\text{JTPid} = (z;q)_\infty (q/z;q)_\infty (q;q)_\infty$$

$$J_{a,m} := j(q^a;q^m) \quad \overline{J}_{a,m} := j(-q^a;q^m)$$

$$J_m := J_{m,3m} = \prod_{i=1}^{\infty} (1-q^{mi})$$

Partial Theta function $\sum_{n=0}^{\infty} (-1)^n z^n q^{\binom{n}{2}}$

False theta function $\sum_{n=-\infty}^{\infty} sg(n)(-1)^n z^n q^{\binom{n}{2}}$

$$sg(n) := \begin{cases} 1 & n \geq 0 \\ -1 & n < 0 \end{cases}$$

$$m(x,q,z) := \frac{1}{j(z;q)} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{\binom{n}{2}} z^n}{1-q^{n-1} xz}$$

modular forms

A modular form is a holomorphic function f on the complex upper-half plane $\mathbb{H} = \{ z \in \mathbb{C} \mid \operatorname{Im} z > 0 \}$ satisfying the transformation equation

$$f\left(\frac{az+b}{cz+d} \right) = (cz+d)^k f(z) \qquad (*)$$

for all $z \in \mathbb{H}$ and all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \leq SL_2(\mathbb{Z})$

— o —

only need to check $(*)$ on generators of Γ

if $\Gamma = SL_2(\mathbb{Z}) = \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\rangle$,

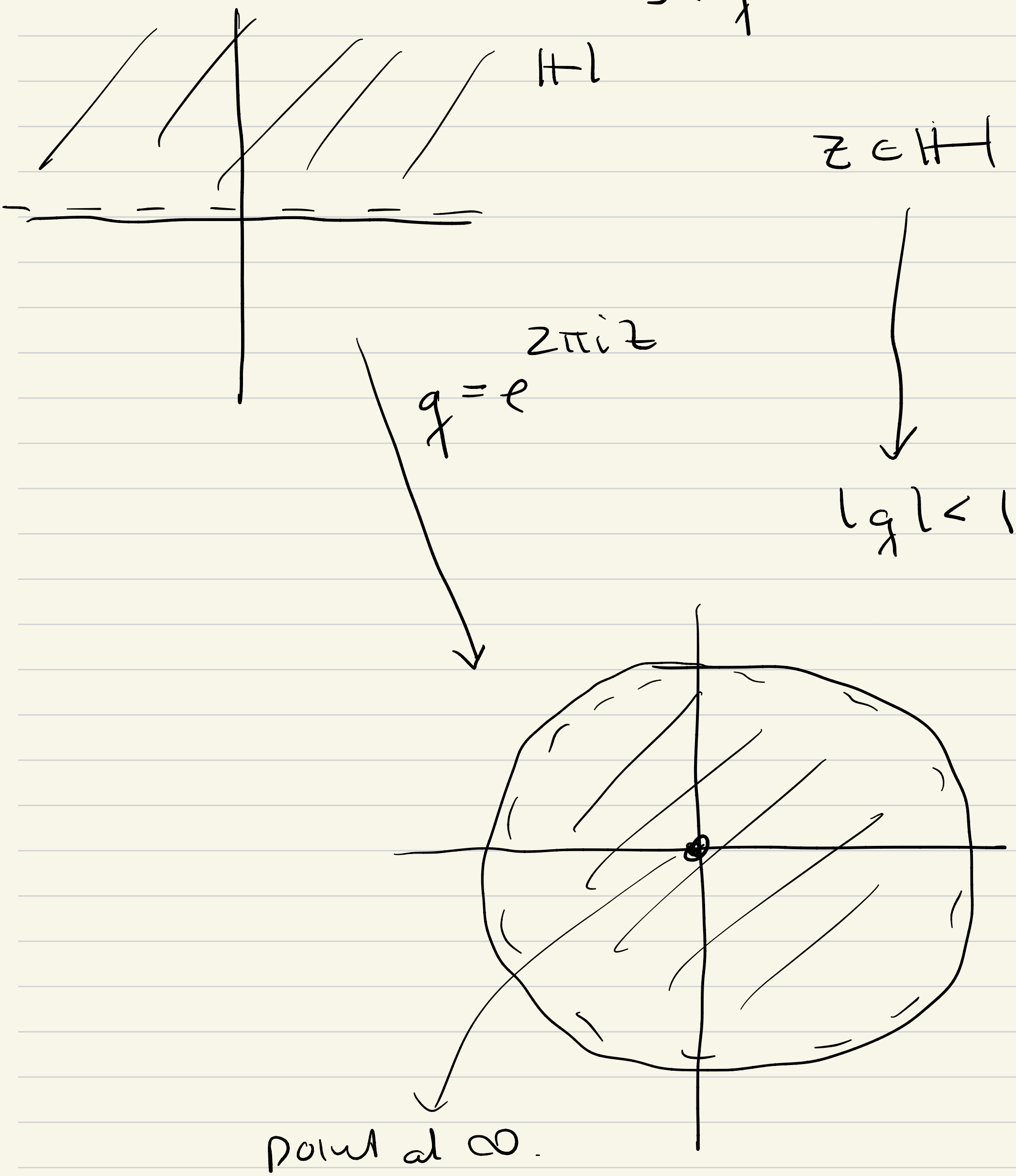
$f(z+1) = f(z)$, $f(-1/z) = z^k f(z)$

Many variations

- Γ, $\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \mid c \equiv 0 \bmod N \right\}$
- $k \in \frac{1}{2}\mathbb{Z}$
- f may be vector-valued
- f may need a correction term

modular forms

$z \in \mathbb{H} := \{ z \in \mathbb{C} \mid \operatorname{Im} z > 0 \}$, $q := e^{2\pi i z}$



modular forms

Example: Eisenstein Series

For even k, $k>2$

$$G_k(\tau) := \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{1}{(m\tau+n)^k} \in M_k(SL_2(\mathbb{Z}))$$

why $k>2$? why even?

$$G_k\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k G_k(\tau)$$

Fourier expansion $q := e^{2\pi i\tau}$ $\tau\in\mathbb{H}$.

$$G_k(\tau) = 2\zeta(k)E_k(\tau)$$

$$E_k(\tau) = 1 - \frac{2k}{B_k}\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n$$

$$\zeta(k) = \sum_{n=1}^{\infty}\frac{1}{n^k} \qquad \sigma_{k-1}(n) = \sum_{0<m|n} m^{k-1}$$

Bernoulli numbers $\displaystyle\frac{t}{e^t-1} = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!}$

modular forms

fact. if $f_1(\tau) \in M_{k_1}(SL_2(\mathbb{Z}))$, $f_2(\tau) \in M_{k_2}(SL_2(\mathbb{Z}))$

then $(f_1 f_2)(\tau) \in M_{k_1+k_2}(SL_2(\mathbb{Z}))$

prove nontrivial identities

$\dim_{\mathbb{C}} M_k(SL_2(\mathbb{Z})) < \infty$

e.g., $\dim_{\mathbb{C}} M_8(SL_2(\mathbb{Z})) = 1$

$$E_4(\tau) := 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n$$

$$E_8(\tau) := 1 + 480 \sum_{n=1}^{\infty} \sigma_7(n) q^n$$

$E_4^2(\tau), E_8(\tau) \in M_8(SL_2(\mathbb{Z}))$

$\Rightarrow E_4^2(\tau) \doteq E_8(\tau)$

$$\Rightarrow \sigma_7(n) = \sigma_3(n) + 120 \sum_{i=1}^{n-1} \sigma_3(i)\sigma_3(n-i) \quad n \geq 1$$

$$\sigma_3(n) = \sum_{0<m|n} m^3 \qquad \sigma_7(n) = \sum_{0<m|n} m^7$$

modular forms

If $f_1, f_2 \in M_k(\Gamma)$

Γ a congruence subgroup

$\Gamma_0(N) \subseteq \Gamma \subseteq SL_2(\mathbb{Z})$

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \,\middle|\, c \equiv 0 \bmod N \right\}$$

Sturm:

need to check $\frac{k}{12}\left[SL_2(\mathbb{Z}) : \Gamma \right]$

Previous page $\Gamma = SL_2(\mathbb{Z})$, $k = 8$

so $\frac{k}{12}\left[SL_2(\mathbb{Z}) : \Gamma \right] = \frac{2}{3}$

Eisenstein Series, $k=2$?

$$G_2(\tau) = \sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}'_m} \frac{1}{(m\tau+n)^2}$$

$\mathbb{Z}'_m = \mathbb{Z} - \{0\}$ if $m=0$, $\mathbb{Z}'_m = \mathbb{Z}$ otherwise

as arranged the terms can be shown to yield

$$G_2(\tau) = 2\zeta(2) - 8\pi^2 \sum_{n=1}^{\infty} \sigma_1(n) q^n$$

$$q = e^{2\pi i \tau} \qquad \sigma_1(n) = \sum_{d|n,\, d>0} d.$$

Fact -

$$G_2\left(\frac{az+b}{cz+d}\right) = (cz+d)^2 G_2(z) - \frac{\pi i c}{cz+d}$$

Hecke's Trick (1-2-3 m.f's, Ono etal)

Consider

$$G_{2,\varepsilon}(z) = \frac{1}{2} \sum_{m,n}{}' \frac{1}{(mz+n)^2 |mz+n|^{2\varepsilon}}$$

$z \in \mathbb{H}$, $\varepsilon > 0$ $\sum'$ means not $(m,n) = (0,0)$

above now converges absolutely

$$G_{2,\varepsilon}\left(\frac{az+b}{cz+d}\right) = (cz+d)^2 |cz+d|^{2\varepsilon} G_{2,\varepsilon}(z)$$

It turns out (much analysis)

$$\lim_{\varepsilon \to 0} G_{2,\varepsilon}(z) = G_2(z) - \frac{\pi}{y}, \quad y = \operatorname{Im}(z)$$

So

$$G_2^*(z) := G_2(z) - \frac{\pi}{y} \qquad y = \operatorname{Im}(z)$$

transforms like a modular form of weight 2

but it is not holomorphic

Hecke's Trick

$$G_2(z) = \sum_{n \neq 0} 1/n^2 + \sum_{m \neq 0} \sum_{n \in \mathbb{Z}} \frac{1}{(mz+n)^2}$$

$$G_2(z) = \pi^2/3 - 8\pi^2 \sum_{n=1}^{\infty} \sigma(n) q^n \qquad q = e^{2\pi i z} \quad z \in \mathbb{H}$$

$$\sigma(n) = \sum_{d \mid n,\ d>0} d$$

G_2 is holomorphic but not modular

$$G_2\left(\frac{az+b}{cz+d}\right) = (cz+d)^2 G_2(z) - \frac{\pi i c}{cz+d}$$

with correction term

$$G_2^*(z) := G_2(z) - \frac{\pi}{y}, \quad y = \operatorname{Im}(z)$$

$G_2^*(z)$ transforms like a modular form of weight two but is not holomorphic.

Mock ϑ fns & real analytic modular forms

Ramanujan's def: A mock ϑ function is a q-hypergeometric series which converges for $0<|q|<1$

where

1) infinitely many roots of unity are exponential singularities
2) for every root of unity ξ there is a ϑ-fn $\vartheta_\xi(q)$ such that $f(q) - \vartheta_\xi(q)$ is bounded as $q \to \xi$ radially
3) there is no ϑ fn that works for all ξ. i.e. f is not the sum of two functions one of which is a ϑ-function and the other a function which is bounded in all roots of unity

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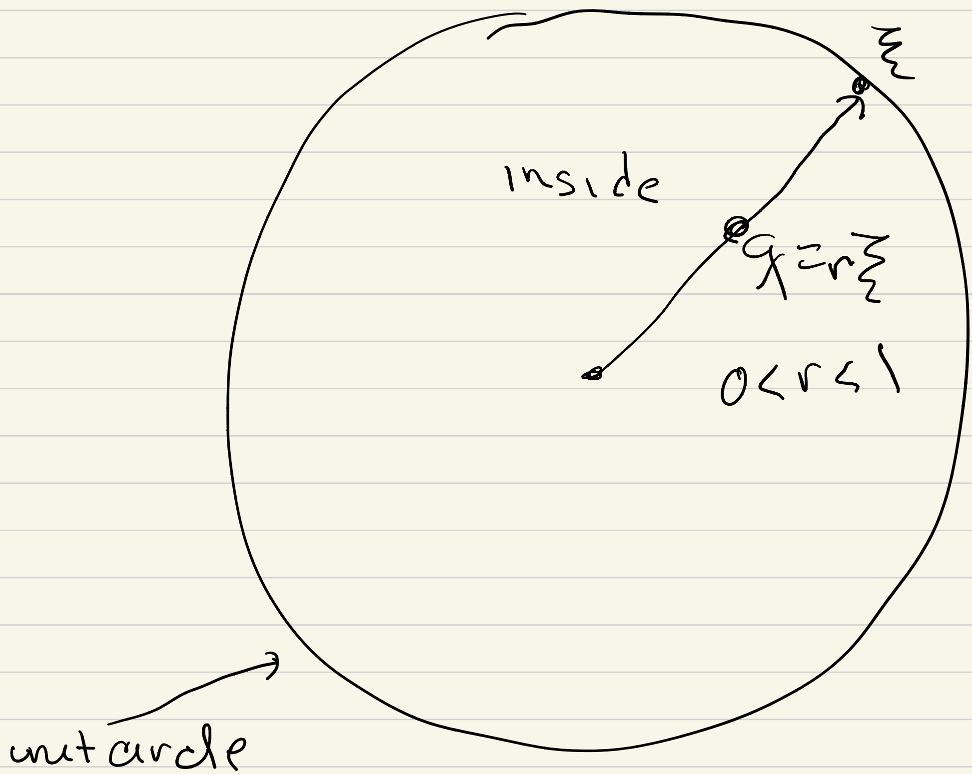
If we relax (3) the definition is not true

Mock ϑ functions & real analytic modular forms

$$f(q) := \sum_{n=0}^{\infty} q^{n^2} \Big/ (1+q)^2(1+q^2)^2 \cdots (1+q^n)^2$$



inside unit circle is $|q| < 1$

ξ is a root of unity say $\xi = e^{2\pi i p/q}$

p, q integers

$q = r\xi$ as $r \to 1^-$ q approaches ξ radially inside unit circle

mock ϑ functions & real analytic modular forms

- Watson → mock theta functions are holomorphic but not modular
- Zwegers → found correction term once a vector-valued mock theta function has been corrected, the new function transforms like a modular form but is no longer holomorphic.
- real analytic modular form
- Bruinier & Funke
  weak Harmonic Maass form.

mock ϑ-functions & real analytic modular forms

Watson & 3rd order mock ϑ functions

$$f(q) := \sum_{n=0}^{\infty} q^{n^2} \Big/ (-q;q)_n^2 \qquad \omega(q) := \sum_{n=0}^{\infty} q^{2n(n+1)} \Big/ (q;q^2)_{n+1}^2$$

$$f_0(\tau) := q^{-\frac{1}{24}} f(q) \quad f_1(\tau) := 2q^{\frac{1}{3}} \omega(q^{\frac{1}{2}}) \quad f_2(\tau) := 2q^{\frac{1}{3}} \omega(-q^{\frac{1}{2}})$$

$$F := (f_0, f_1, f_2)^T \qquad q := e^{2\pi i \tau}$$

$$F(\tau+1) = \begin{pmatrix} \zeta_{24}^{-1} & 0 & 0 \\ 0 & 0 & \zeta_3 \\ 0 & \zeta_3 & 0 \end{pmatrix} F(\tau) \qquad \zeta_n := e^{2\pi i/n}$$

$$\frac{1}{\sqrt{-i\tau}} F(-1/\tau) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} F(\tau) + R(\tau)$$

$$R(\tau) = 4\sqrt{3}\sqrt{-i\tau}\,(j_2(\tau), -j_1(\tau), j_3(\tau))^T$$

where $$j_1(\tau) = \int_0^{\infty} e^{3\pi i \tau x^2} \frac{\sinh 2\pi\tau x}{\sinh 3\pi\tau x}\, dx$$

$j_2(\tau), j_3(\tau)$ similarly.

mock-ϑ functions & real analytic modular forms

Zwegers rewrote $R(\tau)$ in terms of period integrals of certain weight $3/2$ theta functions

$$g_0(z) := \sum_{n\in\mathbb{Z}} (-1)^n (n+1/3)\, e^{3\pi i (n+\frac{1}{3})^2 z}$$

$$g_1(z) := -\sum_{n\in\mathbb{Z}} (n+1/6)\, e^{3\pi i (n+\frac{1}{6})^2 z}$$

$$g_2(z) := \sum_{n\in\mathbb{Z}} (n+1/3)\, e^{3\pi i (n+\frac{1}{3})^2 z}$$

$$\begin{pmatrix} g_0 \\ g_1 \\ g_2 \end{pmatrix}(z+1) = \begin{pmatrix} 0 & 0 & \zeta_6 \\ 0 & \zeta_{24} & 0 \\ \zeta_6 & 0 & 0 \end{pmatrix} \begin{pmatrix} g_0 \\ g_1 \\ g_2 \end{pmatrix}(z) \qquad \zeta_n := e^{2\pi i/n}$$

$$\begin{pmatrix} g_0 \\ g_1 \\ g_2 \end{pmatrix}(-1/z) = -(-iz)^{3/2} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} g_0 \\ g_1 \\ g_2 \end{pmatrix}(z)$$

From transformation properties & Fourier expansions we see that these are <u>cusp forms</u>.

mock ϑ functions & real analytic modular forms

Two Lemmas

Lemma 1: For $\tau \in \mathbb{H}$,

$$R(\tau) = -2i\sqrt{3} \int_0^{i\infty} \frac{g(z)}{\sqrt{-i(z+\tau)}} \, dz$$

where $g(z) = (g_0, g_1, g_2)^T$ and we have to integrate each component of the vector.

-o-

define $$G(\tau) := 2i\sqrt{3} \int_{-\bar{\tau}}^{i\infty} \frac{(g_1(z), g_0(z), -g_2(z))^T}{\sqrt{-i(z+\tau)}} \, dz$$

Lemma 2: For $\tau \in \mathbb{H}$

$$G(\tau+1) = \begin{pmatrix} \zeta_{24}^{-1} & 0 & 0 \\ 0 & 0 & \zeta_3 \\ 0 & \zeta_3 & 0 \end{pmatrix} G(\tau) \qquad \zeta_n := e^{2\pi i/n}$$

$$\frac{1}{\sqrt{-i\tau}} G(-1/\tau) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} G(\tau) + R(\tau)$$

mock ϑ functions & real analytic modular forms

Theorem (Zwegers)

The function $H(z)$ defined by

$$H(z) := F(z) - G(z)$$

is a vector-valued real-analytic modular form of weight $1/2$ satisfying

$$H(\tau+1) = \begin{pmatrix} \zeta_{24}^{-1} & 0 & 0 \\ 0 & 0 & \zeta_3 \\ 0 & \zeta & 0 \end{pmatrix} H(\tau) \qquad \zeta_n := e^{2\pi i/n}$$

$$\frac{1}{\sqrt{-i\tau}} H\left(-\frac{1}{\tau}\right) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} H(\tau)$$

and H is an eigenfunction of the Casimir operator

$$\Omega_{1/2} = -4y^2 \frac{\partial^2}{\partial\tau\,\partial\bar{\tau}} + iy \frac{\partial}{\partial\bar{\tau}} + \frac{3}{16}$$

with eigenvalue $3/16$ where $\tau = x+iy$

$$\frac{\partial}{\partial\tau} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \text{ and } \frac{\partial}{\partial\bar{\tau}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

mock ϑ functions & real analytic modular forms

note that

$$\Omega_{1/2} = -y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)+\frac{iy}{2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)+\frac{3}{16}$$

$$\Omega_{1/2}\, H = \frac{3}{16} H$$

Let's write $\quad F = H + G \qquad (H = F - G)$

Corollary The vector-valued third order mock ϑ function F can be written as the sum of a real analytic modular form H and a function G that is bounded in all rational points.

Remark If Ramanujan's definitions weakened it is not true

Harmonic Maass Forms (Bruinier & Funke)

A harmonic Maass form of weight $k \in \frac{1}{2}\mathbb{Z}$ on $\Gamma_0(N)$ is a smooth function $M: \mathbb{H} \to \mathbb{C}$ satisfying

1) transformation law
2) harmonic condition
3) growth condition

__Remark__

Compare with definition of modular form

__Recall__

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \;\middle|\; c \equiv 0 \bmod N \right\}$$

Harmonic Maass Forms

A harmonic Maass form of weight $k \in \frac{1}{2}\mathbb{Z}$ on $\Gamma_0(N)$ is a smooth function $f : \mathbb{H} \to \mathbb{C}$ satisfying

$\gamma z = \frac{az+b}{cz+d}$

1) transformation law

$\forall \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$ $\tau \in \mathbb{H}$

$$f(\gamma\tau) = \begin{cases} \left(\frac{c}{d}\right)^{2k} \epsilon_d^{-k} (c\tau+d)^k f(\tau) & k \in \frac{1}{2}\mathbb{Z} - \mathbb{Z} \\ (c\tau+d)^k f(\tau) & k \in \mathbb{Z} \end{cases}$$

- $\left(\frac{\cdot}{\cdot}\right)$ Kronecker symbol,
- $\epsilon_d = \begin{cases} 1 & d \equiv 1 \mod 4 \\ i & d \equiv 3 \mod 4 \end{cases}$
- if $k \in \frac{1}{2}\mathbb{Z} - \mathbb{Z}$ then require $4 \mid N$

2) harmonic condition $\Delta_k f = 0$

$$\Delta_k := -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{y^2}\right) + iky\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

$k = 1/2$ gives the real analytic condition of Zwegers

$\tau = x + iy$

Harmonic Maass Forms

A harmonic Maass form of weight $k \in \frac{1}{2}\mathbb{Z}$ on $\Gamma_0(N)$ is a smooth function $M: \mathbb{H} \to \mathbb{C}$ satisfying

3) growth condition

$\exists$ polynomial $P_M \in \mathbb{C}[q^{-1}]$ such that $f(\tau) - P_M(\tau) = O(e^{-\epsilon y})$ for some $\epsilon > 0$ as $y \to \infty$.

$\tau = x + iy$

Notation:

$H_k(\Gamma)$ space of wt k Harmonic Maass forms on Γ

if (3) is replaced w/

(3') $f(\tau) = O(e^{\epsilon y})$ as $y \to \infty$ some $\epsilon > 0$

$H_k^!(\Gamma)$ space of wt k harmonic Maass forms of manageable growth

Harmonic Maass Forms

a harmonic Maass Form f decomposes into two parts

$q := e^{2\pi i \tau}$, $\tau \in \mathbb{H}$, $\tau = x + iy$

$f = f^+ + f^-$

"holomorphic part"

$$f^+ := \sum_{n \geq r_f} c_f^+(n) q^n \qquad r_f \in \mathbb{Z}$$

"non-holomorphic part"

$$f^- := \sum_{n<0} c_f^-(n) \Gamma(1-k, 4\pi|n|y) q^n$$

$$\Gamma(a,x) = \int_x^\infty t^{a-1} e^{-t} \, dt$$

complementary error function
(or incomplete gamma function)

Harmonic Maass forms

terminology

non holomorphic term f^- is a

period integral of the shadow of f^+

shadow is typically a cusp form

if f^+ has wt $2-k$

then shadow has wt k.

Example (Zwegers real analytic modular forms)

$F(\tau) = (f_0, f_1, f_2)^T$

$f_0(\tau) = q^{-1/24} f(q)$, $f_1(\tau) = 2q^{1/3}\omega(q^{1/2})$, $f_2(\tau) = \ldots$

all weight $1/2$

the correction term was a period integral

of weight $3/2$ unary theta functions

that were cusp forms.

Harmonic Maass forms

Examples

$$G_2(z) = \sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}_m'} \frac{1}{(mz+n)^2}$$

$\mathbb{Z}_m' = \mathbb{Z} \setminus \{0\}$ if $m=0$, $\mathbb{Z}_m' = \mathbb{Z}$ otherwise

$$G_2(z) = 2\zeta(2) - 8\pi^2 \sum_{n=1}^{\infty} \sigma(n) q^n, \quad \sigma(n) = \sum_{d \mid n, d>0} d$$

$$E_2(z) = 1 - 24 \sum_{n=1}^{\infty} \sigma(n) q^n$$

$$G_2^*(z) := G_2(z) - \pi/y \qquad z = x+iy.$$

transforms like weight 2 form but is not holomorphic

$$E_2^*(z) = E_2(z) - \frac{3}{\pi y} \qquad \zeta(2) = \frac{\pi^2}{6}$$

$E_2^* \in H_2^!$ weight 2 hmf, manageable growth

with shadow constant modular fn $3/\pi$

E_2^* has wt 2, $3/\pi$ has wt 0.

$2-k = 2 \qquad k = 0$

Harmonic Maass Forms

- where do mock theta functions fit?
- they are "holomorphic parts" of harmonic Maass forms
- Zagier defines a mock modular form to be the holomorphic part of a harmonic Maass form
- Rhoades - Ramanujan's definition of a mock theta function is not equivalent to the modern definition

$$V_1(q) := \frac{1}{(q)_\infty}\left(\frac{1}{12} - \sum_{n \neq 0} \frac{q^n}{(1-q^n)^2}\left(3 + (-1)^n q^{\frac{3n^2-n}{2}}(1+q^n)\right)\right)$$

$$V_2(q) := \frac{1}{(q)_\infty}\left(\frac{1}{12} - 2\sum_{n \neq 0} \frac{nq^n}{(1-q^n)^2}\left(1 + (-1)^{n+1} q^{\binom{n}{2}}\right)\right)$$

Either V_1 modern def but not R's

or V_2 is R's def but not modern

Harmonic Maass Forms

mock theta function

$$f(q) = \sum_{n=0}^{\infty} q^{n^2} \Big/ (-q;q)_n^2$$

$$= 1 + \frac{q}{(1+q)^2} + \frac{q^2}{(1+q)^2(1+q^2)^2} + \cdots$$

$$h(\tau) := q^{-1/24} f(q)$$

$$R(\tau) := -2i\sqrt{3} \int_{-\bar{\tau}}^{i\infty} \frac{g_1(z)}{\sqrt{-i(z+\tau)}}\, dz$$

$$g_1(z) := -\sum_{n \in \mathbb{Z}} \left(n + \tfrac{1}{6}\right) e^{3\pi i \left(n+\frac{1}{6}\right)^2 z}$$

$$= -\frac{1}{6} \sum_{n \equiv 1 \,(\mathrm{mod}\, 6)} n q^{n^2/24}$$

$$= -\frac{1}{6} \sum_{n=1}^{\infty} \left(\frac{-12}{n}\right) n q^{n^2/24}$$

Harmonic Maass Forms

mock theta function

$$f(q) = 1 + \sum_{n=1}^{\infty} q^{n^2} / (-q;q)_n^2$$

$$h(\tau) := q^{-1/24} f(\tau)$$

$$R(\tau) := -2i\sqrt{3} \int_{-\bar{\tau}}^{i\infty} \frac{g_1(z)}{\sqrt{-i(z+\tau)}}\, dz$$

$h(\tau) + R(\tau)$ transforms like modular form wt $1/2$

$$R(\tau) = \sum_{n \equiv 1 \bmod 6} \operatorname{sgn}(n) \beta(n^2 y/6) q^{-n^2/24} \qquad y = \operatorname{Im}(\tau)$$

$$\operatorname{sgn}(n) = \begin{cases} 1 & n \geq 0 \\ -1 & n < 0 \end{cases}$$

$$\beta(x) = \int_x^{\infty} u^{-1/2} e^{-\pi u}\, du = 2\int_{\sqrt{x}}^{\infty} e^{-\pi t^2}\, dt = 1 - E(\sqrt{x})$$

complementary error function

or incomplete gamma function

Harmonic Maass Forms

- identities between modular forms
- to prove two modular forms f_1, f_2 are equal one
  - shows they live in same finite dimensional space k, Γ
  - checks that first few coefficients are equal $\frac{k}{12}[SL_2(\mathbb{Z}) : \Gamma]$

- identities between mock theta functions
- similar idea! but with twists

Harmonic Maass Forms

- identities between mock theta functions
- recall mock theta conjectures
- ten identities relating fifth order mock theta functions to each other
- Andrews & Garvan divided identities into two groups of five identities each
- AG showed that for each group, if one of the five identities is true then so are the other four identities of that group.

Harmonic Maass Forms

- mock theta conjectures
- Hickerson proved the following two identities

$$f_0(q) := \sum_{n=0}^{\infty} q^{n^2} / (-q;q)_n \qquad f_1(q) := \sum_{n=0}^{\infty} q^{n^2+n} / (-q;q)_n$$

$$g(x;q) := -1 + \sum_{n=0}^{\infty} q^{n^2} / (x;q)_{n+1} (q/x;q)_n$$

Theorem (Hickerson)

$$f_0(q) = -2g(q^2;q^{10}) + \bar{J}_5 \bar{J}_{5,10} / \bar{J}_{1,5}$$

$$f_1(q) = -2q^{-1} g(q^4;q^{10}) + \bar{J}_5 \bar{J}_{5,10} / \bar{J}_{2,5}$$

Harmonic Maass Forms

- mock theta conjectures
- Folsom chose two other identities

$$\chi_0(q) := \sum_{n=0}^{\infty} q^n \Big/ (-q^{n+1};q)_n \qquad \chi_1(q) := \sum_{n=0}^{\infty} q^n \Big/ (-q^{n+1};q)_{n+1}$$

$$g(x;q) := -1 + \sum_{n=0}^{\infty} q^{n^2} \Big/ (x;q)_{n+1}(q/x;q)_n$$

$$\chi_0(q) - 1 = 1 + 3q\,g(q,q^5) - J_5^2 J_{2,5} \big/ J_{1,5}^2$$

$$\chi_1(q) = 3q\,g(q^2;q^5) + J_5^2 J_{1,5} \big/ J_{2,5}^2$$

Harmonic Maass Forms

- mock theta conjectures
- Folsom choose two different identities

$$\chi_0(q)-1 = 1+3q\,g(q,q^5) - \bar{J}_5^2 \bar{J}_{2,5}/\bar{J}_{1,5}$$

$$\chi_1(q) = 3q\,g(q,q^5) + \bar{J}_5^2 \bar{J}_{1,5}/\bar{J}_{2,5}$$

roughly (lots of details)

- complete both sides by finding the appropriate non holomorphic terms
- show both sides transform like weight ½ form on $\Gamma = \Gamma_1(144 \cdot 10^2 \cdot 5^4)$
- show first few coeffs $\frac{k}{12}[SL_2(\mathbb{Z}) : \Gamma]$ are equal
- show non-holomorphic terms cancel
- gives equality for the holomorphic parts

Harmonic Maass forms

mod 2 theta conjectures

how to compute $\frac{k}{12}\left[SL_2(\mathbb{Z}) : \Gamma\right]$

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \;\middle|\; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod N \right\}$$

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \;\middle|\; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \bmod N \right\}$$

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \;\middle|\; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \bmod N \right\}$$

$$\left[SL_2(\mathbb{Z}) : \Gamma(N)\right] = N^3 \prod_{p \mid N} \left(1 - \frac{1}{p^2}\right)$$

$$\left[\Gamma_1(N) : \Gamma(N)\right] = N$$

$$\left[SL_2(\mathbb{Z}) : \Gamma_1(N)\right] = N^2 \prod_{p \mid N} \left(1 - \frac{1}{p^2}\right)$$

$N = 144 \cdot 10^2 \cdot 5^4$ p's $= 2, 3, 5$, $k/12 = 1/24$

$$\frac{N^2}{24} \prod_{p \mid N} \left(1 - \frac{1}{p^2}\right) = 2.16 \times 10^{12} \text{ coeffs}$$

Andersen $\rightsquigarrow$ 16 coeffs

Hurwitz class numbers

Review terminology

- Let $f(x,y) := ax^2 + bxy + cy^2$ be a binary quadratic form. The discriminant is defined $D(f) := b^2 - 4ac$

- A positive definite quadratic form (a,b,c) of disc. D is said to be reduced if $|b| \leq a \leq c$ and if in addition when one of the two inequalities is an equality we have $b \geq 0$.

- A reduced form is primitive if $\gcd(a,b,c) = 1$ and imprimitive otherwise.

- The number of reduced forms of a given discriminant D is finite in number

Hurwitz class numbers

Review terminology

- We define the class number $h(D)$ to be the number of primitive positive definite reduced quadratic forms of discriminant D.

- Let N be a non-negative. The Hurwitz class number $H(N)$ is defined

1) If $N \equiv 1, 2 \pmod 4$ then $H(N) = 0$

2) If $N = 0$ then $H(0) = -1/12$

3) If $N > 0$, $N \equiv 0, 3 \pmod 4$ then $H(N)$ is the class number of positive definite binary quadratic forms of disc. $-N$ with those classes that contain a multiple of x^2+y^2 or x^2+xy+y^2 counted with weight $1/2$ or $1/3$ respectively

Hurwitz class numbers

Review terminology

Hecke pointed out (Gauss & Hermite)

$$r_3(N) = \begin{cases} 12H(N) & \text{if } N \equiv 1, 2 \bmod 4 \\ 24H(N) & \text{if } N \equiv 3 \bmod 8 \\ 0 & \text{if } N \equiv 7 \bmod 8 \\ r_3(N/4) & \text{if } N \equiv 0 \bmod 4 \end{cases}$$

Python Code!

1) compute $r_3(n)$ & compare OEIS
2) compute $H(n)$ & compare OEIS
3) check above formula for $r_3(n)$

$r_3(n)$ = # of ways to write n as sum of 3 □'s.

$$\sum_{n=0}^{\infty} r_3(n) q^n = \left(\sum_{n \in \mathbb{Z}} q^{n^2}\right)^3$$

Hurwitz class numbers

Review terminology

Hecke pointed out (Gauss & Hermite)

$$r_3(N) = \begin{cases} 12 H(N) & \text{if } N \equiv 1, 2 \bmod 4 \\ 24 H(N) & \text{if } N \equiv 3 \bmod 8 \\ 0 & \text{if } N \equiv 7 \bmod 8 \\ r_3(N/4) & \text{if } N \equiv 0 \bmod 4 \end{cases}$$

where $r_3(N)$ is the number of representations of N as a sum of three squares

$$\sum_{N=0}^{\infty} r_3(N) q^N = \theta^3, \quad \theta = \sum_{t \in \mathbb{Z}} q^{t^2} = 1 + 2q + 2q^4 + \cdots$$

Hecke suggested

$$\mathcal{H}(z) := \sum_{N=0}^{\infty} H(N) q^N, \quad q = e^{2\pi i z}, \quad z \in \mathbb{H}$$

should be a modular form of weight $3/2$

Hurwitz Class Numbers

Hecke's Suggestion

$$\mathcal{H}(z) := \sum_{N=0}^{\infty} H(N) q^N, \quad q := e^{2\pi i z} \quad z \in \mathbb{H}$$

should be a modular form of weight 3/2

- at the time no satisfactory theory of weight one-half modular forms (Shimura)
- still didn't transform like a weight 3/2 modular form
- generalizations of H(N) were shown to be modular (Cohen)
- $\mathcal{H}(z)$ can be completed to a function which transforms under $\Gamma_0(4)$ like a weight 3/2 form (Zagier)

Hurwitz Class Numbers

Generalizations (Cohen)

For $k > 0$ even $M_k(\Gamma_0(D), \chi_D)$

denotes the vector space of modular forms

of weight k, level D, and nebentypus χ_D.

$$f\left(\frac{az+b}{cz+d}\right) = \chi_D(a)(cz+d)^k f(z) \qquad z \in \mathbb{H}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(D) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \;\middle|\; c \equiv 0 \bmod N \right\}$$

Hurwitz class numbers

generalizations (Cohen)

$H(r,N)$ $\quad r$ positive odd, $N \in \mathbb{N}$

$H(r,0) = \zeta(1-2r)$

$H(r,N) = 0 \quad N \equiv 1,2 \bmod 4$

$$H(r,N) = L(1-r,\chi_\Delta) \sum_{d \mid f} d^{2r-1} \prod_{p \mid d} (1 - \chi_\Delta(p) p^{-s})$$

for $N > 0$, $N \equiv 0,3 \bmod 4$

$\Delta < 0$ is disc of $\mathbb{Q}(\sqrt{-N})$

f is defined $-N = \Delta f^2$

$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} \quad L(s,\chi) = \sum_{n=1}^{\infty} \chi(n) n^{-s}$

- generalizes $H(N) = H(1,N)$
- $H_D(r,N) := \sum_{t \in \mathbb{Z}} H\left(r, \frac{4N-t^2}{D}\right)$
- $\sum_{N=0}^{\infty} H_D(r,N) q^N \in M_{r+1}(\Gamma_0(D), \chi_D) \quad r > 1.$

Hurwitz class numbers

Harmonic Maass Forms

(Zagier) For $z \in \mathbb{H}$,

$$\mathcal{F}(z) := \sum_{n=0}^{\infty} H(n) q^n + y^{-1/2} \sum_{f=-\infty}^{\infty} \beta(4\pi f^2 y) q^{-f^2}$$

where $y = \operatorname{Im} z$, $q := e^{2\pi i z}$, $\beta(x)$ defined

$$\beta(x) = \frac{1}{16\pi} \int_1^{\infty} u^{-3/2} e^{-xu} \, du \qquad x \geq 0$$

transforms under $\Gamma_0(4)$ like wt $3/2$ modular form

Hurwitz Class numbers

Harmonic Maass forms

$$\mathcal{H}(z) := -\frac{1}{12} + \sum_{n=1}^{\infty} H(n) q^n + \frac{1}{4\sqrt{\pi}} \sum_{n=1}^{\infty} n \Gamma\left(-\frac{1}{2}, 4\pi n^2 y\right) q^{-n^2}$$

$$+ \frac{1}{8\pi y} \quad \text{(Ono's dal book)}$$

where $z = x+iy$, is a weight $3/2$ harmonic Maass form of manageable growth on $\Gamma_0(4)$.

Remarks

- $\Gamma(s,z) = \int_z^{\infty} e^{-t} t^{s-1} \, dt$
- $\mathcal{H}^+(z) := -\frac{1}{12} + \sum_{n=1}^{\infty} H(n) q^n$ is the Eisenstein–Hurwitz mock modular form of $3/2$
- Shadow of $\mathcal{H}^+(z)$ is $-\frac{1}{16}\Theta$, $\Theta = \sum_{n \in \mathbb{Z}} q^{n^2}$

Summary

- modular forms & eisenstein series
- real analytic modular forms & mock ϑ functions
- harmonic Maass forms
  - definitions & terminology
  - examples
    - E_2
    - mocktheta functions
    - Hurwitz class numbers (Python)
  - proving identities

    mock theta functions

Next time?

tbd.

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