Lecture 7: 3 November 2020

1) q-hypergeometric series & the transformation $q \to q^{-1}$
   a) Rogers-Ramanujan identities
   b) mock theta functions & partial theta functions

2) mock ϑ functions & realanalytic modular forms
   a) a conjecture of Zwegers
   b) mock modularity

Lecture 7: 3 November 2020

1) The transformation $q \to q^{-1}$

a) Andrews & Baxter's solution to the hard hexagon model of statistical mechanics

b) dual nature between mock theta functions and partial theta functions

2) mock theta functions & real analytic modular forms (Zwegers)

a) a mock theta function, a false theta function, radial asymptotics & an conjecture of Zwegers

b) mock modularity of third order mock theta functions

Notation $\quad q \in \mathbb{C}, \quad 0<|q|<1$

$$(x)_\infty = (x;q)_\infty = \prod_{i=0}^{\infty} (1-q^i x)$$

$$j(z;q) = \sum_{n=-\infty}^{\infty} (-1)^n z^n q^{\binom{n}{2}}$$

$$\text{JTPid} \quad = (z;q)_\infty (q/z;q)_\infty (q;q)_\infty$$

$$J_{a,m} := j(q^a;q^m) \qquad \overline{J}_{a,m} := j(-q^a;q^m)$$

$$J_m := J_{m,3m} = \prod_{i=1}^{\infty} (1-q^{mi})$$

Partial Theta function $\displaystyle\sum_{n=0}^{\infty} (-1)^n z^n q^{\binom{n}{2}}$

False theta function $\displaystyle\sum_{n=-\infty}^{\infty} sg(n)(-1)^n z^n q^{\binom{n}{2}}$

$$sg(n) := \begin{cases} 1 & n \geq 0 \\ -1 & n < 0 \end{cases}$$

$$m(x,q,z) := \frac{1}{j(z;q)} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{\binom{n}{2}} z^n}{1-q^{n-1} xz}$$

Andrews & Baxter's solution to hard hexagon model

- The transformation $q \to q^{-1}$ is useful in problems in which the theory of partitions is applied to statistical mechanics
- Although $q \to q^{-1}$ may make sense for a q-hypergeometric series it will not for a product

Example $\quad \nu(q) := \sum_{n=0}^{\infty} q^{n(n+1)} \Big/ (-q;q^2)_{n+1}$

$\nu(q)$ converges for $|q|<1$, $|q|>1$

replace q with $1/q$

fact: $(a;q^{-1})_n = (a^{-1};q)_n (-a)^n q^{-\binom{n}{2}}$

$$\nu(q) \xrightarrow[q \to q^{-1}]{} \sum_{n=0}^{\infty} q^{-n(n+1)} \Big/ (-q^{-1};q^{-2})_{n+1}$$

$$= \sum_{n=0}^{\infty} q^{n+1} \Big/ (-q;q^2)_{n+1} =: \nu_-(q)$$

Andrews & Baxter's solution to hard hexagon model

- Baxter found a solution to the hard hexagon model of statistical mechanics
- his solution divided the model into four regimes I, II, III, IV.
- his solutions to regimes I, III, IV eight known Rogers-Ramanujan identities
- for regime II he conjectured six multi-sum Rogers-Ramanujan like identities

Andrews & Baxter's solution to hard hexagon model

- Baxter's eight Rogers-Ramanujan identities are all either in Rogers's work or immediate consequences of it.

Region I

a) $\sum_{n=0}^{\infty} q^{n^2}/(q;q)_n = J_5 / J_{1,5}$

b) $\sum_{n=0}^{\infty} q^{n^2+n}/(q;q)_n = J_5 / J_{2,5}$

Region IV?

a) $\sum_{n=0}^{\infty} q^{n(3n-1)/2} \Big/ (q;q)_n (q;q^2)_n = J_{4,10} / J_1$

b) $\sum_{n=0}^{\infty} q^{3n(n+1)/2} \Big/ (q;q)_n (q;q^2)_{n+1} = J_{2,10} / J_1$

Region IV

a) $\sum_{n=0}^{\infty} q^{n(n+1)} / (q;q)_{2n+1} = J_{3,10} J_{4,20} \Big/ J_1 J_{20}$

b) $\sum_{n=0}^{\infty} q^{n(n+1)} / (q;q)_{2n} = J_{1,10} J_{8,20} \Big/ J_1 J_{20}$

+ four more identities

Andrews & Baxter's solution to hard hexagon model

Regime III (Conjectures by Baxter / Theorems by Andrews)

Theorem 1

$$\sum_{n=0}^{\infty} \sum_{0 \le r \le (3n+1)/2} \frac{q^{(3n^2+3n)/2 - r}}{(q^2;q^2)_r (q;q)_{3n-2r+1}} = \frac{1}{J_1}\left(J_{4,15} + q J_{1,15} \right)$$

Theorem 3

$$\sum_{n=1}^{\infty} \sum_{0 \le r \le (3n-1)/2} \frac{q^{n(3n-1)/2 - r}}{(q^2;q^2)_r (q;q)_{3n-2r-1}} = J_{6,15} / J_1$$

+ four more

Andrews & Baxter's solution to hard hexagon model

- In resolving Baxter's conjectures, Andrews established a $q \to q^{-1}$ duality between Regions II & III and Regions I & IV.

- what do we mean?

- Andrews used finite versions which converge to infinite q-series in the limit.

- To translate between Regions II & III and prove the conjectures for Region II, Andrews used identities which have origins in work of Schur.

Andrews & Baxter's solution to hard hexagon model

- Translate between Regions II & III
- identities with origin in Schur (two)

$$
\text{a)} \sum_{n,r\geq 0} q^{n(3n+1)/2} \begin{bmatrix} N-2n-2r \\ n \end{bmatrix}_q \begin{bmatrix} r+n \\ r \end{bmatrix}_{q^2} q^{r}
$$

$$
= \sum_{\lambda=-\infty}^{\infty} (-1)^{\lambda} q^{5\lambda^2+\lambda} \begin{bmatrix} N \\ \left[\frac{N-5\lambda}{2}\right] \end{bmatrix}_q
$$

$$
\text{where } \begin{bmatrix} N \\ M \end{bmatrix}_q = \begin{cases} (q)_N \big/ (q)_{N-M}(q)_M & M\geq 0 \\ 0 & M<0, N\geq 0 \end{cases}
$$

$[x]$ is largest integer not exceeding x.

Andrews & Baxter's solution to hard hexagon model

- Translate between Regions II & III
- To obtain the first Rogers-Ramanujan identity of Region III, let $N \to \infty$ in (a)

- To obtain three of the six identities in Region II
  - replace N with $3N+k$, $k=-1,0,1$
  - replace q with q^{-1}
  - let $N \to \infty$

useful facts

$$(a;q^{-1})_n = (a^{-1};q)_n (-a)^n q^{-\binom{n}{2}}$$

q-binomial theorem

$$\sum_{r=0}^{\infty} \frac{(a;q)_n z^n}{(q;q)_n} = \frac{(az;q)_\infty}{(z;q)_\infty}$$

The transformation $q \to q^{-1}$:

mock theta functions & partial theta functions

Recall our heuristic

$$m(x,q,z) \sim \sum_{r=0}^{\infty} (-1)^r x^r q^{-\binom{r+1}{2}}$$

$$(a;q^{-1})_n = (a^{-1};q)_n \, (-a)^n q^{-\binom{n}{2}}$$

2nd order

$$B(q) = \sum_{n=0}^{\infty} q^n \frac{(-q;q^2)_n}{(q;q^2)_{n+1}} = -q^{-1} m(1, q^4, q^3)$$

$$\sim -q^{-1} \sum_{n=0}^{\infty} (-1)^n q^{-4\binom{n+1}{2}}$$

$$q \to q^{-1} \Big\downarrow \qquad\qquad\qquad \Big\downarrow q \to q^{-1}$$

$$\sum_{n=0}^{\infty} (-1)^{n+1} \frac{q^{n+1} (-q;q^2)_n}{(q;q^2)_{n+1}} = -q \sum_{r=0}^{\infty} (-1)^r q^{4\binom{r+1}{2}}$$

numerically we discover equality

proof from Bailey's Lemma

The transformation $q \to q^{-1}$:

mock theta functions & partial theta functions

Recall our heuristic

$$m(x, q, z) \sim \sum_{r=0}^{\infty} (-1)^r x^r q^{-\binom{r+1}{2}}$$

$$(a; q^{-1})_n = (a^{-1}; q)_n \, (-a)^n q^{-\binom{n}{2}}$$

But we have a second form for $B(q)$

$$B(q) = \sum_{n=0}^{\infty} q^{n^2+n} \frac{(-q^2; q^2)_n}{(q; q^2)^2_{n+1}} = -q^{-1} m(1, q^4, q^3)$$

this time $q \to q^{-1}$ yields a different...

$$\sum_{n=0}^{\infty} q^{2(n+1)} \frac{(-q^2; q^2)_n}{(q; q^2)^2_{n+1}}$$

$$= -q \sum_{n=0}^{\infty} (-1)^n q^{4\binom{n+1}{2}} + q \frac{J_4}{J_{1,2}} \sum_{n=-\infty}^{\infty} sg(n) q^{-n} q^{3n(n+1)}$$

new, very mysterious

The transformation $q \to q^{-1}$:

mock theta functions & partial theta functions

Recall our heuristic

$$m(x,q,z) \sim \sum_{r=0}^{\infty} (-1)^r x^r q^{-\binom{r+1}{2}}$$

$$(a;q^{-1})_n = (a^{-1};q)_n (-a)^n q^{-\binom{n}{2}}$$

$$B(q) = \sum_{n=0}^{\infty} q^n \frac{(-q;q^2)_n}{(q;q^2)_{n+1}} = \sum_{n=0}^{\infty} q^{n^2+n} \frac{(-q;q^2)_n}{(q;q^2)^2_{n+1}} = -q^{-1} m(1,q^4,q^3)$$

$q \to q^{-1}$

$$\sum_{n=0}^{\infty} (-1)^{n+1} q^{n+1} \frac{(-q;q^2)_n}{(q;q^2)_{n+1}} = -q \sum_{n=0}^{\infty} (-1)^n q^{2n(n+1)}$$

$$\sum_{n=0}^{\infty} q^{2(n+1)} \frac{(-q^2;q^2)_n}{(q;q^2)^2_{n+1}} = -q \sum_{n=0}^{\infty} (-1)^n q^{2n(n+1)} + q \frac{J_4}{J_{1,2}} \sum_{n=-\infty}^{\infty} sg(n) q^{3n^2+2n}$$

$q \to q^{-1}$

The transformation $q \to q^{-1}$:

mock theta functions & partial theta functions

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Four tenth order mock theta functions

and their six identities (Choi & RLN)

$$\phi(q) := \sum_{n=0}^{\infty} q^{\binom{n+1}{2}} \Big/ (q;q^2)_{n+1} \qquad \psi(q) := \sum_{n=0}^{\infty} q^{\binom{n+2}{2}} \Big/ (q;q^2)_{n+1}$$

& two more

$$q^{2}\phi(q^{9}) - \left(\frac{\psi(\omega q) - \psi(\omega^{2} q)}{\omega - \omega^{2}}\right) = -q \frac{\bar{J}_{1,2}}{\bar{J}_{3,6}} \cdot \frac{\bar{J}_{3,15} \bar{J}_{6}}{\bar{J}_{3}}$$

$$q^{-2}\psi(q^{9}) + \left(\frac{\omega\phi(\omega q) - \omega^{2}\phi(\omega^{2} q)}{\omega - \omega^{2}}\right) = \frac{\bar{J}_{1,2}}{\bar{J}_{3,6}} \cdot \frac{\bar{J}_{6,15} \bar{J}_{6}}{\bar{J}_{3}}$$

+ four more $\qquad 1+\omega+\omega^{2}=0$

what happens with $q \to q^{-1}$?

The transformation $q \to q^{-1}$:

mock theta functions & partial theta functions

Recall our heuristic

$$m(x, q, z) \sim \sum_{r=0}^{\infty} (-1)^r x^r q^{-\binom{r+1}{2}}$$

$$(a; q^{-1})_n = (a^{-1}; q)_n \, (-a)^n q^{-\binom{n}{2}}$$

$$\phi(q) := \sum_{n=0}^{\infty} \frac{q^{\binom{n+1}{2}}}{(q; q^2)_{n+1}} = -q^{-1} m(q, q^{10}, q) - q^{-1} m(q, q^{10}, q^2)$$

trick $\quad m(x, q, z) = x^{-1} m(x^{-1}, q, z^{-1})$

$$\phi(q) := \sum_{n=0}^{\infty} \cdots = -q^{-1} m(q, q^{10}, q) - q^{-2} m(q^{-1}, q^{10}, q^2)$$

$\downarrow$ heuristic

$$\sim -q^{-1} \sum_{r=0}^{\infty} (-1)^r q^r q^{-10\binom{r+1}{2}} - q^{-2} \sum_{r=0}^{\infty} (-1)^r q^{-r} q^{10\binom{r+1}{2}}$$

$q \to q^{-1}$ $\qquad\qquad \downarrow q \to q^{-1}$

$$\phi_D(q) := \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n+2}{2}}}{(q; q^2)_{n+1}} = -q \sum_{r=0}^{\infty} (-1)^r q^{5r(r+1)} - q^2 \sum_{r=0}^{\infty} (-1)^r q^r q^{5r(r+1)}$$

$\hookrightarrow$ Bailey.

The transformation $q \to q^{-1}$:

mock theta functions & partial theta functions

Recall our heuristic

$$m(x,q,z) \sim \sum_{r=0}^{\infty} (-1)^r x^r q^{-\binom{r+1}{2}}$$

$$(a;q^{-1})_n = (a^{-1};q)_n (-a)^n q^{-\binom{n}{2}}$$

Similary

$$\psi(q) = \sum_{n=0}^{\infty} \frac{q^{\binom{n+2}{2}}}{(q;q^2)_{n+1}} = -m(q^3,q^{10},q) - m(q^3,q^{10},q^3)$$

$$= -m(q^3,q^{10},q) - q^{-3} m(q^{-3},q^{10},q^{-3})$$

$\downarrow$ heuristic

$q \to q^{-1}$

$$\sim -\sum_{r=0}^{\infty} (-1)^r q^{3r} q^{-10\binom{r+1}{2}} - q^{-3}\sum_{r=0}^{\infty} (-1)^r q^{-3r} q^{-10\binom{r+1}{2}}$$

$\downarrow$ $q \to q^{-1}$

$$\psi_0(q) := \sum_{n=0}^{\infty} (-1)^{n+1} \frac{q^{\binom{n+1}{2}}}{(q;q^2)_{n+1}} = -\sum_{r=0}^{\infty} (-1)^r q^{-3r} q^{5r(r+1)} - q^3 \sum_{r=0}^{\infty} (-1)^r q^{3r} q^{5r(r+1)}$$

Bailey's lemma

The transformation $q \to q^{-1}$:

mock theta functions & partial theta functions

— o —

Four tenth order mock theta functions

and their six identities (Choi & RLN)

$$\phi(q) := \sum_{n=0}^{\infty} q^{\binom{n+1}{2}} \Big/ (q;q^2)_{n+1} \qquad \psi(q) := \sum_{n=0}^{\infty} q^{\binom{n+2}{2}} \Big/ (q;q^2)_{n+1}$$

$1 + \omega + \omega^2 = 0$

$$q^2 \phi(q^9) - \left(\frac{\psi(\omega q) - \psi(\omega^2 q)}{\omega - \omega^2} \right) = -q \frac{\bar{J}_{1,2}}{\bar{J}_{3,6}} \cdot \frac{\bar{J}_{3,15} \bar{J}_6}{\bar{J}_3}$$

$$q^{-2} \psi(q^9) + \left(\frac{\omega \phi(\omega q) - \omega^2 \phi(\omega^2 q)}{\omega - \omega^2} \right) = \frac{\bar{J}_{1,2}}{\bar{J}_{3,6}} \cdot \frac{\bar{J}_{6,15} \bar{J}_6}{\bar{J}_3}$$

$$q^{-2} \phi_D(q^9) - \left(\frac{\psi_D(\omega^2 q) - \psi_D(\omega q)}{\omega - \omega^2} \right) = 0$$

$$q^2 \psi_D(q^9) - \left(\frac{\omega \phi_D(\omega^2 q) - \omega^2 \phi_D(\omega q)}{\omega - \omega^2} \right) = 0$$

+ four more $= 0$

The transformation $q \to q^{-1}$:

mock theta functions & partial theta functions

Recall our heuristic

$$m(x, q, z) \sim \sum_{r=0}^{\infty} (-1)^r x^r q^{-\binom{r+1}{2}}.$$

Question 1: (2^{nd} order $B(q)$) why do two different q-hypergeometric series yielding same $m(x,q,z)$ yield similar partial theta fns?

Question 2: why do the dual identities of the six identities for the four 10^{th} order mock theta functions all equal zero?

Are there finite versions which allow us to translate from one set to another?

Mock ϑ fns & real analytic modular forms

Ramanujan's def: A mock ϑ function is a q-hypergeometric series which converges for $0<|q|<1$

where

1) infinitely many roots of unity are exponential singularities

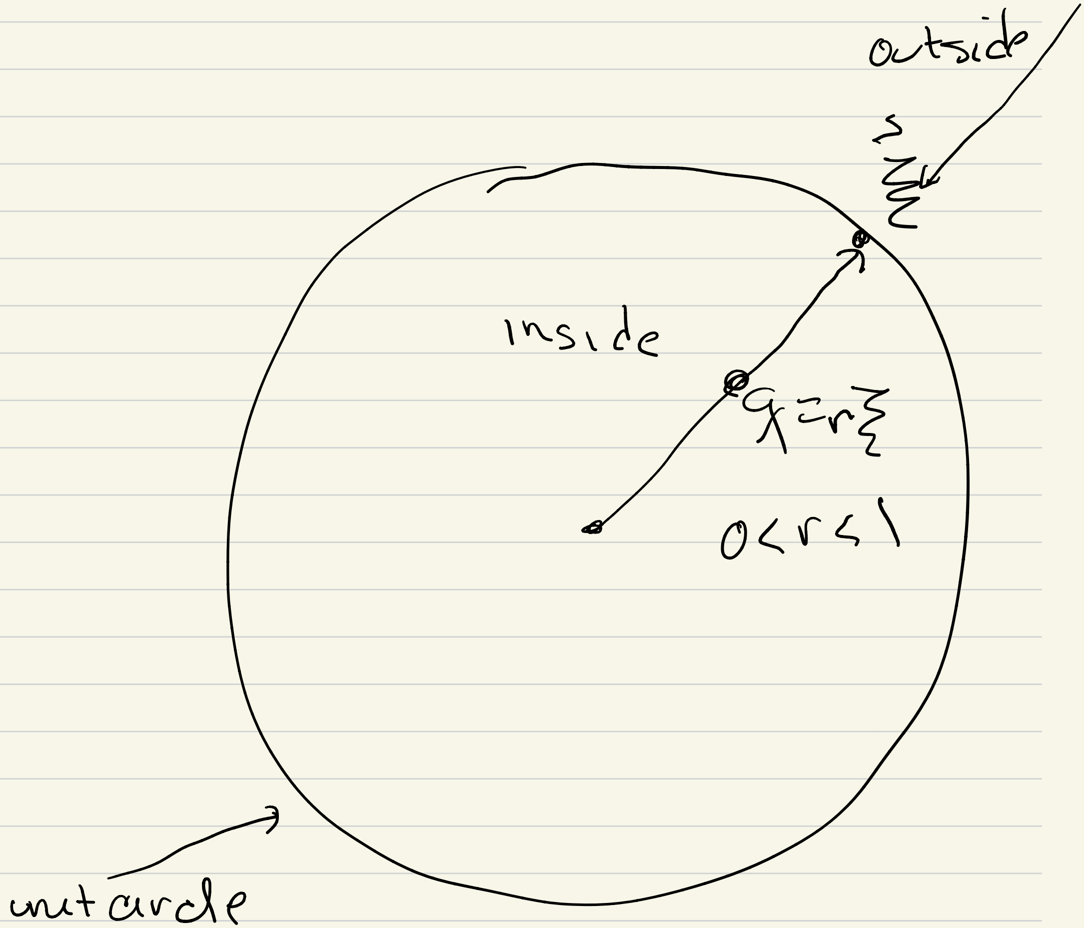
2) for every root of unity ξ there is a ϑ-fn $\vartheta_\xi(q)$ such that $f(q) - \vartheta_\xi(q)$ is bounded as $q \to \xi$ radially

3) there is no ϑ fn that works for all ξ. i.e. f is not the sum of two functions one of which is a ϑ-function and the other a function which is bounded in all roots of unity

(1),(2) Watson & Selberg

(3) Griffin, Ono, Rolen

Mock ϑ functions & real analytic modular forms



inside unit circle is $|q|<1$

ξ is a root of unity say $\xi = e^{2\pi i p/q}$

p, q integers

$q = r\xi$ as $r \to 1^-$ q approaches ξ radially inside unit circle

mock ϑ-functions & real analytic modular forms

Let us consider the third order mock ϑ-function

$$\nu(q) := \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(-q;q^2)_{n+1}}$$

$$= \frac{1}{1+q} + \frac{q^2}{(1+q)(1+q^3)} + \frac{q^6}{(1+q)(1+q^3)(1+q^5)} + \cdots$$

$\nu(q)$ converges for $|q|<1$ (ratio test)

but also for $|q|>1$. To see this replace

$q \to q^{-1}$, $|q|<1$

$$\nu_-(q) = \sum_{n=0}^{\infty} q^{-n(n+1)} \Big/ (-q^{-1};q^{-2})_{n+1}$$

$$= \sum_{n=0}^{\infty} q^{n+1} \Big/ (-q;q^2)_{n+1}$$

$$(a;q^{-1})_n = (a^{-1};q)_n (-a)^n q^{-\binom{n}{2}}$$

mock ϑ functions & real analytic modular forms

we can also write $\nu_-(q)$ in terms of partial theta functions (and in this case a false ϑ-function)

Recall heuristic $m(x,q,z) \sim \sum_{r=0}^{\infty} (-1)^r x^r q^{-\binom{r+1}{2}}$

$$\nu(q) := \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(-q;q^2)_{n+1}} = q^{-1} m(q^2, q^{12}, -q^3) - q^{-1} m(q^2, q^{12}, -q^9)$$

$$m(x,q,z) = x^{-1} m(x^{-1}, q, z^{-1})$$

$$= q^{-1} m(q^2, q^{12}, -q^3) - q^{-3} m(q^{-2}, q^{12}, -q^{-9})$$

$$\sim q^{-1} \sum_{r=0}^{\infty} (-1)^r q^{2r} q^{-6r(r+1)} - q^{-3} \sum_{r=0}^{\infty} (-1)^r q^{-2r} q^{6r(r+1)}$$

$q \to q^{-1}$ $\qquad\qquad\qquad\qquad \downarrow q \to q^{-1}$

$$\nu_-(q) := \sum_{n=0}^{\infty} q^{n+1} \Big/ (-q;q^2)_{n+1} = q \sum_{r=0}^{\infty} (-1)^r q^{-2r} q^{6r(r+1)} - q^3 \sum_{r=0}^{\infty} (-1)^r q^{2r} q^{6r(r+1)}$$

$\to$ Id in RLN

mock ϑ functions & real analytic modular forms

$$\nu_-(q) = \sum_{r=0}^{\infty} (-1)^r q^{6r^2+4r+1} (1+q^{4r+2})$$

$$* = \sum_{n=-\infty}^{\infty} sg(n)(-1)^n q^{6n^2+4n+1} \qquad sg(n) := \begin{cases} 1 & n \geq 0 \\ -1 & n < 0 \end{cases}$$

$$= q^{1/3} \sum_{n=0}^{\infty} (-1)^{n+1} \left(\frac{-3}{n}\right) q^{\frac{2}{3}n^2} \qquad \left(\frac{-3}{n}\right) = \begin{cases} 1 & n \equiv 1 \ (3) \\ -1 & n \equiv 2 \ (3) \\ 0 & n \equiv 0 \ (3) \end{cases}$$

$*$ looks like a theta function

$$j(z;q) = \sum_{n=-\infty}^{\infty} (-1)^n z^n q^{\binom{n}{2}}$$

but the signs are different

Rogers used the term false theta function

Q: how do we compare? ξ root of unity

$\nu(q)$ $q \to \xi$ radially inside?

& $\nu_-(q)$ $q \to \xi$ radially outside?

mock ϑ functions & real analytic modular forms

$$\nu(q) := \sum_{n=0}^{\infty} q^{n(n+1)} \Big/ (1+q)(1+q^3)\cdots(1+q^{2n+1})$$

if $q \to \xi$ $\quad \xi$ root of unity $o(\xi) = 2k$, k odd

then $\lim_{q\to\xi} \nu(q)$ is unbounded

but by Ramanujan's claim we should be able to cut out the singularities with ϑ functions

for other ξ, $\lim_{q\to\xi} \nu(q)$ is bounded

but what about $\nu_-(q)$?

mock ϑ functions & real analytic modular forms

radial limits for $\nu_-(q)$

Can compute complete asymptotic expansion

Lawrence, Zagier

Let $C : \mathbb{Z} \to \mathbb{C}$ be a periodic function with mean zero.

Then the associated L-series $L(s, C) := \sum_{n=1}^{\infty} C(n) n^{-s}$

$(\mathrm{Re}(s) > 1)$ extends holomorphically to $\mathbb{C}$. The

two functions have the asymptotic expansions: $t > 0$

$$\sum_{n=1}^{\infty} C(n) e^{-nt} \sim \sum_{r=0}^{\infty} L(-r, C) \, (-t)^r / r!$$

$$\sum_{n=1}^{\infty} C(n) e^{-n^2 t} \sim \sum_{r=0}^{\infty} L(-2r, C) \, (-t)^2 / r!$$

as $t \searrow 0$.

—o—

To get the asymptotic expansion of $\nu_-(q)$

as $q \to \xi$ radially write $q = \xi e^{-t}$.

mock ϑ functions & real analytic modular forms

the false theta function becomes,

$$\nu_-(q) := q^{\frac{1}{3}} \sum_{n=0}^{\infty} (-1)^{n+1} \left(\frac{-3}{n}\right) q^{\frac{2}{3}n^2}$$

$$= \xi^{1/3} e^{-\frac{1}{3}t} \sum_{n=0}^{\infty} (-1)^{n+1} \left(\frac{-3}{n}\right) \xi^{\frac{2}{3}n^2} e^{-\frac{2}{3}n^2 t}$$

we want to conside $t \searrow 0$

$$C(n) := (-1)^{n+1} \left(\frac{-3}{n}\right) \xi^{\frac{2}{3}n^2}$$

we check $C(n)$ is a periodic function w/ mean zero

if K is order of ξ then C has period $6K$

& $C(6K-n) = C(n) \Rightarrow$ mean is zero.

—o—

- So LZ result describes completely the behaviour of ν outside the unit circle

- is behaviour of ν outside unit circle related to behaviour of ν inside unit circle?

mock ϑ functions & real analytic modular forms

Conjecture (Zwegers)

a) If ξ is a root of unity where ν is bounded (as $q \to \xi$ radially inside unit circle), e.g. $\xi = 1$, then ν is C^∞ over a line radially thru ξ.

b) If ξ is a root of unity where ν is not bounded e.g. $\xi = -1$, then the asymptotic expansion of the bounded term (2),

where bounded term is $\nu(q) - \theta_\xi(q)$, is the same as the asymptotic expansion of ν as $q \to \xi$ radially outside the unit circle.

—o—

let's do some numerical work for (a)

mock ϑ functions & real analytic modular forms

Conjecture (Zwegers)

numerical work for part (a)

Q1) If ξ root of unity $\lim_{q \to \xi} \nu(q)$ bounded

as $q \to \xi$ radially how sporadic are the values?

$o(\xi) \leq 100$ Python plot

Q2) If ξ root of unity $\lim_{q \to \xi} \nu(q)$ bounded

as $q \to \xi$ radially does the curve look C^∞

as one passes from inside the unit circle to outside the unit circle? Python plot

Remark: here unbounded behaviour occurs

when $o(\xi) = 2k$, k odd so for

Q1 & Q2 $o(\xi) \not\equiv 2 \bmod 4$.

sporadic on boundary but smooth on radial line

mock ϑ functions & real analytic modular forms

Conjecture (Zwegers)

numerical work for part (a)

Q3) If ξ root of unity $\lim_{q \to \xi} \nu(q)$ bounded

$q \to \xi$ radially how does

$\lim_{q \to \xi} \nu(q)$ compare to behavior of $\nu_-(q)$

where $\nu_-(q)$ has asymptotics, $q := \xi e^{-t}$

$$\nu_-(q) = \xi e^{\frac{1}{3} - \frac{1}{3}t} \sum_{n=0}^{\infty} (-1)^{n+1} \left(\frac{-3}{n}\right) \xi^{\frac{2}{3}n^2} e^{-\frac{2}{3}n^2 t}$$

$$t \searrow 0 \qquad \sim \xi^{\frac{1}{3}} e^{-\frac{1}{3}t} \sum_{r=0}^{\infty} L(-2r, C) \left(-\tfrac{2}{3}t\right)^r / r!$$

$$C(n) := (-1)^{n+1} \left(\frac{-3}{n}\right) \xi^{\frac{2}{3}n^2} e^{-\frac{2}{3}t^2}, \quad L(s, C) := \sum_{n=0}^{\infty} C(n) n^{-s}$$

Answer: Python examples suggest equality!!

note: inside $q \to \xi$ outside use $\bar{\xi}$ conjugate.

mock ϑ functions & real analytic modular forms

Conjecture (Zwegers)

@31 (Computational Remark)

note that for ξ where $q \to \xi$ radially

$\lim_{q \to \xi} \nu(q)$ bounded we can write

$$o(\xi) = 4k, 4k+1, 4k+3$$

$$\sum_{n=0}^{\infty} \xi^{n(n+1)} \Big/ (-\xi;\xi^2)_{n+1}$$

$$= \frac{1}{1-R} \sum_{n=0}^{4k-1} \xi^{n(n+1)} \Big/ (-\xi;\xi^2)_{n+1}$$

$$R = \frac{1}{(-\xi;\xi^2)_{4k}}$$

mock ϑ functions & real analytic modular forms

Conjecture (Zwegers)

b) If ξ is a root of unity ν is not bounded, e.g. $\xi=-1$, then the asymptotic expansion of the bounded term $\nu(q)-\Theta_\xi(q)$ (in R's def) is the same as the asymptotic expansion of ν as $q\to\xi$ radially outside the unit circle

—o—

for a ξ, $o(\xi)=2k$, k odd like (a)

how does $\lim_{q\to\xi} \nu(q)-\Theta_\xi(q)$ compare to

$$t\searrow \quad \xi^{1/3}e^{-1/3 t}\sum_{r=0}^{\infty} L(-2r,c)\,(-2/3\,t)^r/r!$$

$$C(n)=(-1)^{n+1}\xi^{2/3 n^2}e^{-2/3\,tn^2}, \quad L(s,c)=\sum_{n=0}^{\infty}C(n)n^{-s}$$

- First find $\Theta_\xi(q)$ then do numerical work.

mock ϑ fns & real analytic modular forms

Radial Limits for $\nu(q)$, unbounded roots

$$\nu(q) := \sum_{n=0}^{\infty} q^{n(n+1)} / (-q;q^2)_{n+1} = g_3(i\sqrt{q}\,; q)$$

universal mock ϑ function

$$g_3(x;q) := \sum_{n=0}^{\infty} q^{n(n+1)} / (x;q)_{n+1} (q/x;q)_{n+1}$$

From Lost Notebook (like Zudilin's proof)

$$R_2(\omega;q) + U_2(\omega;q) = -\omega \frac{j(\omega q;q^2)}{J_2} m(\omega^2, q^2, \omega^{-1} q)$$

$$R_2(\omega;q) := \sum_{n=0}^{\infty} q^{2n^2+2n+1} / (\omega q, q/\omega; q^2)_{n+1}$$

$$U_2(\omega;q) := \sum_{n=0}^{\infty} q^{2n+1} (\omega q, q/\omega; q^2)_n$$

Like Zudilin's proof we want to rewrite $m(\omega^2, q^2, \omega^{-1} q)$

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Radial limits for $\nu(q)$

$$R_2(\omega;q) + U_2(\omega;q) = -\omega \frac{j(\omega q;q^2)}{J_2} m(\omega^2, q, \omega^{-1} q)$$

$$R_2(\omega;q) := \sum_{n=0}^{\infty} q^{2n^2+2n+1} \Big/ (\omega q, q/\omega; q^2)_{n+1}$$

$$U_2(\omega;q) := \sum_{n=0}^{\infty} q^{2n+1} (\omega q, q/\omega; q^2)_n$$

$\omega \to i$, $i^2 = -1$

$$q\nu(q^2) + \sum_{n=0}^{\infty} q^{2n+1} (-q^2;q^4)_n$$

$$= -i \frac{j(iq;q^2)}{J_2} m(-1, q, -iq)$$

Like Zudilin's proof we want to rewrite

here it is much easier!

To Do: Compare

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Radial limits for $\nu(q)$

Recall changing-z theorem

$$m(x,q,z_1) = m(x,q,z_0) + \frac{z_0 J_1^3\, j(z_1/z_0;q)\, j(x z_0 z_1;q)}{j(z_0;q)\, j(z_1;q)\, j(x z_0;q)\, j(x z_1;q)}$$

Recall $\quad m(q,q^2,-1) = 1/2$

$\qquad\qquad m(-1,q^2,q) = 0$

$$m(-1,q^2,-iq) = \underbrace{m(-1,q^2,q)}_{0} + \frac{q J_2^3\, j(-i;q^2)\, j(iq;q^2)}{j(q;q^2)\, j(-q;q^2)\, j(-iq;q^2)\, j(iq;q^2)}$$

simplify with product rearrangements

$$m(-1,q^2,-iq) = \frac{q(1+i)^2 J_2^3 J_{2,8}^2}{J_{2,4}^2 J_{4,8}^2}$$

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Radial limits for $\nu(q)$

Product rearrangements

$$j(-i;q^2) = (-i,q^2)_\infty (iq^2;q^2)_\infty (q^2;q^2)_\infty$$

$$= (1+i)(-iq^2;q^2)_\infty (iq^2;q^2)_\infty (q^2;q^2)_\infty$$

$$= (1+i)(-q^4;q^4)_\infty (q^2;q^2)_\infty$$

$$= (1+i)\, J_{2,8}$$

$$j(iq;q^2) = (iq;q^2)_\infty (-iq;q^2)_\infty (q^2;q^2)_\infty$$

$$= (-q^2;q^4)_\infty (q^2;q^2)_\infty$$

$$= \frac{(-q^2;q^2)_\infty}{(-q^4;q^4)_\infty} (q^2;q^2)_\infty$$

$$= J_{4,8}$$

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Radial limits for $\nu(q)$

back to our bilateral q-series

$$q\nu(q^2) + \sum_{n=0}^{\infty} q^{2n+1}(-q^2;q^4)_n$$

$$= -i\,\frac{\overline{j}(iq;q^2)}{J_2}\, m(-1,q,-iq)$$

$$= -i(1+i)^2 q \cdot \frac{\overline{J}_{4,8}}{J_2}\,\frac{J_2^3\,\overline{J}_{2,8}^2}{\overline{J}_{2,4}^2\,\overline{J}_{4,8}^2}$$

product rearrangements

$\overline{J}_{1,2} = \overline{J}_1^2 / J_2$, $\overline{J}_{1,4} = \overline{J}_1 J_4 / J_2$

$$\nu(q^2) + \sum_{n=0}^{\infty} q^{2n}(-q^2;q^4)_n = 2\,\overline{J}_8^3 / J_4^2$$

$$\Big\downarrow\ q \to q^{1/2}$$

$$\nu(q) + \sum_{n=0}^{\infty} q^{n}(-q;q^2)_n = 2\,\overline{J}_4^3 / J_2^2$$

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Radial limits for $\nu(q)$

From previous page

$$\nu(q) := \sum_{n=0}^{\infty} q^{n(n+1)} \Big/ (1+q)(1+q^3)\cdots(1+q^{2n+1})$$

$$\nu(q) + \sum_{n=0}^{\infty} q^n (-q;q^2)_n = 2 J_4^3 / J_2^2$$

rearranging ξ root of unity order $2k$, k odd

$$\lim_{q \to \xi} \left(\nu(q) - 2 J_4^3 / J_2^2 \right) = \sum_{n=0}^{\frac{k-1}{2}} \xi^n (-\xi;\xi^2)_n$$

compare with Zudilin's proof

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b) Conjecture

If ξ is a root of unity where ν is not bounded, e.g. $\xi = -1$,

(or in general ξ root of order $2k$, k odd)

then the asymptotic expansion of

$\nu(q) - 2J_4^3/J_2^2$ $\quad q \to \xi$ radially inside circle

is the same as $\nu(q)$ as $q \to \xi$ radially outside

$$\xi^{1/3} e^{-1/3 t} \sum_{n=0}^{\infty} (-1)^{n+1} \left(\frac{-3}{n}\right) \xi^{2/3 n^2} e^{-2/3 t n^2}$$

$$\sim \xi^{1/3} e^{-1/3 t} \sum_{r=0}^{\infty} L(-2r, C) \left(-2/3\, t\right)^r / r! \qquad \text{as } t \searrow 0$$

$$L(s, C) = \sum_{n=0}^{\infty} C(n) n^{-s} \qquad C(n) := (-1)^{n+1} \left(\frac{-3}{n}\right) \xi^{2/3 n^2} e^{-2/3 t n^2}$$

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b) Conjecture

$$\lim_{q\to\xi} \sigma(q) - 2J_4^3/J_2^2 = \sum_{n=0}^{\frac{k-1}{2}} \xi^n(-\xi;\xi^2)_n$$

$o(\xi)=2k$, k odd

a simplified version of conjecture is

for ξ root of unity $o(\xi)=2k$, k odd

$$\sum_{n=0}^{\frac{k-1}{2}} \xi^n(-\xi;\xi^2)_n \qquad \text{(LZ proposition)}$$

$$= \lim_{t\searrow 0} \xi^{-1/3} e^{-\frac{1}{3}t} \sum_{r=0}^{\infty} L(-2r, C)(-^2/_3 t)^r/r!$$

$$L(s,C) = \sum_{n=0}^{\infty} C(n) n^{-s} \qquad C(n) := -(-1)^{n+1}\left(\frac{-3}{n}\right)\xi^{-2/3 n^2} e^{-2/3 t n^2}$$

Python work!

1) how sporadic is $\sum_{n=0}^{\frac{k-1}{2}} \xi^n(-\xi;\xi^2)_n$?

$o(\xi)=2k$, k odd $\quad 2k \leq 100$

2) Does equality hold for the first few roots of unity?

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Conjecture (Zwegers)

a) If ξ is a root of unity where ϑ is bounded (as $q \to \xi$ radially inside unit circle), e.g. $\xi = 1$, then ϑ is C^∞ over a line radially thru ξ.

b) If ξ is a root of unity where ϑ is not bounded e.g. $\xi = -1$, then the asymptotic expansion of the bounded term (2), where bounded term is $\vartheta(q) - \theta_\xi(q)$, is the same as the asymptotic expansion of ϑ as $q \to \xi$ radially outside the unit circle.

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easy consequences of the conjecture

do check numerically

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Conjecture (Zwegers)

Consequences

$$C(n) := (-1)^{n+1}\left(\frac{-3}{n}\right)\xi^{-\frac{2}{3}n^2} e^{-\frac{2}{3}tn^2}, \quad L(s,C) := \sum_{n=0}^{\infty} C(n)\, n^{-s}$$

$$\xi^{-\frac{1}{3}} e^{-\frac{1}{3}t} \sum_{r=0}^{\infty} L(-2r, C)\frac{\left(-\frac{2}{3}t\right)^r}{r!} = \begin{cases} \nu(\xi) & o(\xi) \not\equiv 2 \bmod 4 \\ \lim_{q\to\xi} \nu(q) - 2\bar{J}_4^3/\bar{J}_2^2 \\ \quad = \sum_{n=0}^{\frac{k-1}{2}} \xi^n (-\xi;\xi^2)_n & o(\xi) = 2k,\ k \text{ odd} \end{cases}$$

$\lim t \searrow 0$

Recall Python plots demonstrating sporadic nature of $\nu(\xi)$ $o(\xi) \not\equiv 2 \pmod 4$

& $\sum_{n=0}^{\frac{k-1}{2}} \xi^n (-\xi;\xi^2)_n$ $o(\xi) = 2k$, k odd

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Conjecture (Zwegers)

assume conjecture & compare to Ramanujan's def.

Suppose we have a function $\tilde{\nu}$

defined in & outside unit circle and at roots of unity

a) $\tilde{\nu}$ is holomorphic in & outside unit circle

b) $\tilde{\nu}$ C^∞ on all radial lines thru roots of unity

c) $\tilde{\nu} = \nu$ outside unit circle

consequences

a) $\nu - \tilde{\nu} = 0$ outside unit circle

b) $q \to \xi$, $\nu(\xi)$ bdd, $\nu - \tilde{\nu}$ has asymp. exp zero

c) $q \to \xi$, $\nu(\xi)$ unbdd, $\nu - \tilde{\nu} - 2J_4^2/J_2^2$ has asymp exp zero

this suggests $\nu - \tilde{\nu}$ is modular

if so, $\nu = \nu - \tilde{\nu} + \tilde{\nu}$, $\nu - \tilde{\nu}$ theta, $\tilde{\nu}$ bdd all roots of 1.

which contradicts Ramanujan's def

mock ϑ-functions & real analytic modular forms

A modular form is a holomorphic function f on the complex upper-half plane $\mathbb{H} = \{ z \in \mathbb{C} \mid \operatorname{Im} z > 0 \}$ satisfying the transformation equation

$$f\left(\frac{az+b}{cz+d} \right) = (cz+d)^k f(z) \qquad (*)$$

for all $z \in \mathbb{H}$ and all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \subseteq SL_2(\mathbb{Z})$

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many variations

- $k \in \frac{1}{2}\mathbb{Z}$
- $\Gamma = SL_2(\mathbb{Z})$, $\Gamma = \Gamma_0(N) = \left\{ \gamma \in SL_2(\mathbb{Z}) \,\middle|\, \gamma \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \bmod N \right\}$
- f could be vector-valued
- meromorphic instead of holomorphic
- f may need correction term

Ex: Eisenstein series

Theta functions

$\rightarrow$ only need to check $(*)$ on generators of Γ

mock ϑ-functions & real analytic modular forms

Watson & 3rd order mock ϑ functions

$$f(q) := \sum_{n=0}^{\infty} q^{n^2} \Big/ (-q;q)_n^2 \qquad \omega(q) := \sum_{n=0}^{\infty} q^{2n(n+1)} \Big/ (q;q^2)_{n+1}^2$$

$$f_0(\tau) := q^{-\frac{1}{24}} f(q) \quad f_1(\tau) := 2q^{\frac{1}{3}} \omega(q^{\frac{1}{2}}) \quad f_2(\tau) := 2q^{\frac{1}{3}} \omega(-q^{\frac{1}{2}})$$

$$F := (f_0, f_1, f_2)^T \qquad q := e^{2\pi i \tau}$$

$$F(\tau+1) = \begin{pmatrix} \zeta_{24}^{-1} & 0 & 0 \\ 0 & 0 & \zeta_3 \\ 0 & \zeta_3 & 0 \end{pmatrix} F(\tau) \qquad \zeta_n := e^{2\pi i/n}$$

$$\frac{1}{\sqrt{-i\tau}} F(-1/\tau) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} F(\tau) + R(\tau)$$

$$R(\tau) = 4\sqrt{3}\sqrt{-i\tau}\, (j_2(\tau), -j_1(\tau), j_3(\tau))^T$$

where $j_1(\tau) = \displaystyle\int_0^\infty e^{3\pi i \tau x^2} \frac{\sinh 2\pi\tau x}{\sinh 3\pi\tau x}\, dx$

$j_2(\tau), j_3(\tau)$ similarly.

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Zwegers rewrote $R(\tau)$ in terms of period integrals of certain weight $3/2$ theta functions

$$g_0(z) := \sum_{n\in\mathbb{Z}} (-1)^n (n+1/3)\, e^{3\pi i (n+\frac{1}{3})^2 z}$$

$$g_1(z) := -\sum_{n\in\mathbb{Z}} (n+1/6)\, e^{3\pi i (n+\frac{1}{6})^2 z}$$

$$g_2(z) := \sum_{n\in\mathbb{Z}} (n+1/3)\, e^{3\pi i (n+\frac{1}{3})^2 z}$$

$$\begin{pmatrix} g_0 \\ g_1 \\ g_2 \end{pmatrix}(z+1) = \begin{pmatrix} 0 & 0 & \zeta_6 \\ 0 & \zeta_{24} & 0 \\ \zeta_6 & 0 & 0 \end{pmatrix} \begin{pmatrix} g_0 \\ g_1 \\ g_2 \end{pmatrix}(z) \qquad \zeta_n := e^{2\pi i/n}$$

$$\begin{pmatrix} g_0 \\ g_1 \\ g_2 \end{pmatrix}(-1/z) = -(-iz)^{3/2} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} g_0 \\ g_1 \\ g_2 \end{pmatrix}(z)$$

From transformation properties & Fourier expansions we see that these are <u>cusp forms</u>.

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Two Lemmas

Lemma 1: For $\tau \in \mathbb{H}$,

$$R(\tau) = -2i\sqrt{3}\int_0^{i\infty} \frac{g(z)}{\sqrt{-i(z+\tau)}}\,dz$$

where $g(z) = (g_0, g_1, g_2)^T$ and we have to integrate each component of the vector.

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define $$G(\tau) := 2i\sqrt{3}\int_{-\bar{\tau}}^{i\infty} \frac{(g_1(z), g_0(z), -g_2(z))^T}{\sqrt{-i(z+\tau)}}\,dz$$

Lemma 2: For $\tau \in \mathbb{H}$

$$G(\tau+1) = \begin{pmatrix} \zeta_{24}^{-1} & 0 & 0 \\ 0 & 0 & \zeta_3 \\ 0 & \zeta_3 & 0 \end{pmatrix} G(\tau) \qquad \zeta_n := e^{2\pi i/n}$$

$$\frac{1}{\sqrt{-i\tau}}\, G(-1/\tau) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} G(\tau) + R(\tau)$$

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Proofs of Lemmas

Basic analysis from Whittaker & Watson

good exercise to work through & understand

- theory of partial fraction decompositions
- interchange order of integration & summation
- partial integration
- Abel's theorem on continuity

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Theorem (Zwegers)

The function $H(\tau)$ defined by

$$H(\tau) := F(\tau) - G(\tau)$$

is a vector-valued real-analytic modular form

of weight $1/2$ satisfying

$$H(\tau+1) = \begin{pmatrix} \zeta_{24}^{-1} & 0 & 0 \\ 0 & 0 & \zeta_3 \\ 0 & \zeta_3 & 0 \end{pmatrix} H(\tau) \qquad \zeta_n := e^{2\pi i/n}$$

$$\frac{1}{\sqrt{-i\tau}} H\left(-\frac{1}{\tau}\right) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} H(\tau)$$

and H is an eigenfunction of the Casimir operator

$$\Omega_{1/2} = -4y^2 \frac{\partial^2}{\partial\tau\,\partial\bar\tau} + iy\frac{\partial}{\partial\bar\tau} + \frac{3}{16}$$

with eigenvalue $3/16$ where $\tau = x+iy$

$$\frac{\partial}{\partial\tau} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \text{ and } \frac{\partial}{\partial\bar\tau} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

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note that

$$\Omega_{1/2} = -y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)+\frac{iy}{2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)+\frac{3}{16}$$

$$\Omega_{1/2} H = \frac{3}{16} H$$

Let's write $\quad F = H + G \qquad (H = F - G)$

<u>Corollary</u> The vector-valued third order mock ϑ function F can be written as the sum of a real analytic modular form H and a function G that is bounded in all rational points.

<u>Remark</u> If Ramanujan's definitions weakened it is not true

Summary

① the transformation $q \to q^{-1}$

a) Rogers-Ramanujan type identities

b) mock theta functions &

partial theta functions

② asymptotics as $q \to \xi$ root of unit radially

a) mock theta function $\nu(q)$

b) false theta function $\nu_-(q)$

c) Python work

③ introduction to mock modularity

a) real analytic modular forms

Next time:

modularity

mode modularity

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