How Ramanujan May Have Discovered the Mock Theta Functions

$$\Rightarrow \sum_{n\geq 0} \frac{(a;q)_n t^n}{(q;q)_n} = \frac{(at;q)_\infty}{(t;q)_\infty}$$

$$(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})$$

special cases

$$\Rightarrow \sum_{n\geq 0} \frac{t^n}{(q;q)_n} = \frac{1}{(t;q)_\infty}$$

$$\sum_{n\geq 0} \frac{t^n q^{\binom{n}{2}}}{(q;q)_n} = (-t;q)_\infty .$$

Heine

$$\phi\begin{pmatrix} a, b; q, t \\ c \end{pmatrix}$$

$$= \frac{(b;q)_\infty (at;q)_\infty}{(c;q)_\infty (t;q)_\infty} \phi\begin{pmatrix} c/b, t; q, b \\ at \end{pmatrix}$$

where

$$\phi\begin{pmatrix} a, b; t, q \\ c \end{pmatrix} =$$

$$= \sum (a;q)_n (b;q)_n t^n$$

$$\Rightarrow \sum_{n\geq 0} \frac{(a;q)_n (b;q)_n}{(q;q)_n (c;q)_n}$$

$$\phi\begin{pmatrix} a\ b \\ c \end{pmatrix}; q, t$$

$$= \sum_{n\geq 0} \frac{(a;q)_n (b;q)_n t^n}{(q;q)_n (c;q)_n}$$

$$\Rightarrow \left(\text{note } (b;q)_n = \frac{(b;q)_\infty}{(bq^n;q)_\infty}\right)$$

$$= \frac{(b;q)_\infty}{(c;q)_\infty} \sum_{n\geq 0} \frac{(a;q)_n t^n}{(q;q)_n} \frac{(cq^n;q)_\infty}{(bq^n;q)_\infty}$$

$$= \frac{(b;q)_\infty}{(c;q)_\infty} \sum_{n\geq 0} \frac{(a;q)_n t^n}{(q;q)_n} \sum_{m\geq 0} \frac{(c/b;q)_m b^m q^{nm}}{(q;q)_m}$$

$$= \frac{(b;q)_\infty}{(c;q)_\infty} \sum_{m\geq 0} \frac{(c/b;q)_m b^m}{(q;q)_m} \frac{(atq^m;q)_\infty}{(tq^m;q)_\infty}$$

$$= \frac{(b;q)(at;q)_\infty}{(c;q)_\infty (t;q)_\infty} \sum_{m\geq 0} \frac{(c/b;q)_m (t;q)_m b^m}{(q;q)_m (at;q)_m}.$$

Jacobi

$$\Rightarrow \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n^2} = \frac{1}{(q;q)_\infty}.$$

$$= \sum_{n\geq 0} p(n) q^n.$$

$$= \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q^2;q^2)_n} \cdot \frac{(-q;q)_n}{(q;q)_n}$$

$$= \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q^2;q^2)_n} \frac{(-q;q)_\infty}{(q;q)_\infty} \frac{(q^{n+1};q)_\infty}{(-q^{n+1};q)_\infty}$$

$$= \frac{(-q;q)_\infty}{(q;q)_\infty} \sum_{n\geq 0} \frac{q^{n}}{(q^2;q^2)_n} \sum_{m\geq 0} \frac{(-1;q)_m (-q^{n})^{m}}{(q;q)_m}$$

$$= \frac{(-q;q)_\infty}{(q;q)_\infty} \sum_{m\geq 0} \frac{(-1;q)_m (-q)^m}{(q;q)_m} (-q^{m+1};q^2)_\infty ,$$

$$= \frac{(-q;q)_\infty}{(q;q)_\infty} \left\{ \sum_{m\geq 0} \frac{(-1;q)_{2m} q^{2m}}{(q;q)_{2m}} (-q^{2m+1};q^2)_\infty \right.$$

$$\left. - \sum_{m\geq 0} \frac{(-1;q)_{2m+1} q^{2m+1} (-q^{2m+2};q^2)_\infty}{(q;q)_{2m+1}} \right\}$$

$$= \frac{(-q;q)_\infty}{(q;q)_\infty} \left\{ (-q;q^2)_\infty \sum_{m\geq 0} \frac{(-1;q)_{2m} q^{2m}}{(q;q)_{2m} (-q;q^2)_m} \right.$$

$$- (-q^2;q^2)_\infty \sum_{m\geq 0} \frac{(-1;q)_{2m+1} q^{2m+1}}{(q;q)_{2m+1} (-q^2;q^2)_m}$$

$$\Big((-1;q) \quad = (-1;q^2) \quad (-q;q^2)$$

$$(-1;q)_{2m} = (-1;q^2)_m (-q;q^2)_m$$

$$(-1;q)_{2m+1} = 2(-q^2;q^2)_m(-q;q^2)_m$$

$$= \frac{(-q;q)_\infty}{(q;q)_\infty}\left\{(-q;q^2)_\infty \boxed{\sum_{m\geq 0}\frac{(-1;q^2)_m\, q^{2m}}{(q^2;q^2)_m(q;q^2)_m}} - 2(-q^2;q^2)_\infty \sum_{m\geq 0}\frac{(-q;q^2)_m\, q^{2m+1}}{(q;q^2)_{m+1}(q^2;q^2)_m}\right\}$$

Let us consider

$$\sum_{m\geq 0}\frac{(-1;q^2)_m\, q^{2m}}{(q^2;q^2)_m(+q;q^2)_m}$$

$$= \frac{1}{(-q;q^2)_\infty}\sum_{m\geq 0}\frac{(-1;q^2)_m\, q^{2m}}{(q^2;q^2)_m}(+q^{2m+1};q^2)_\infty$$

$$= \frac{1}{(-q;q^2)_\infty}\sum_{m\geq 0}\frac{(-1;q^2)_m\, q^{2m}}{(q^2;q^2)_m}\sum_{n\geq 0}\frac{q^{n^2+2mn}(-1)^n}{(q^2;q^2)_n}$$

$$= \frac{1}{(-q;q^2)_\infty} \sum_{n\geq 0} \frac{q^{n^2}(-1)^n}{(q^2;q^2)_n} \frac{(+q^{2n+2};q^2)_\infty}{(q^{2n+2};q^2)_\infty}$$

$$= \frac{(+q^2;q^2)_\infty}{(-q;q^2)_\infty (q^2;q^2)_\infty} \sum_{n\geq 0} \frac{q^{n^2}(-1)^n}{(-q^2;q^2)_n}$$

A similar result happens for the second series, the resulting infinite series is

$$\sum_{n\geq 1} \frac{(-1)^n q^{n^2}}{(-q;q^2)_n}$$

Once this is all put together

$$\frac{(q;q)_\infty}{(-q;q)_\infty^2} = \phi_3(-q) + 2\psi_3(-q)$$

where

$$\phi_3(q) = \sum_{n\geq 0} \frac{q^{n^2}}{(-q^2;q^2)_n}$$

$$\psi_3(q) = \sum_{n\geq 1} \frac{q^{n^2}}{(q;q^2)_n}$$

I believe that this was the AHA moment for Ramanujan.

$$\frac{(q;q)_\infty}{(-q;q)_\infty^2} = \sum_{n\geq 0} \frac{(-1)^n q^{n^2}}{(-q^2;q^2)_n}$$

$$+2\sum_{n\geq 1}\frac{(-1)\,q}{(-q;q^2)_n}$$

Consider behavior of RHS near $q=-1$.

$$f_0(q)=\sum_{n=0}^{\infty}\frac{q^{n^2}}{(-q;q)_n^2}$$

(B) Rogers–Ramanujan

$$\sum_{n\geq 0}\frac{q^{n^2}}{(q;q)_n}=\frac{1}{(q,q^4;q^5)_\infty}$$

5th order mock theta

$$\sum_{n\geq 0}\frac{q^{n^2}}{(-q;q)_n}$$

$n \geq 0$

$$\frac{(q;q)_\infty}{(zq;q)_\infty (q/z;q)_\infty}$$

Lost Notebook

$(z=-1)$

$\Rightarrow p(5n+4) \equiv 0 \pmod 5$

$\Rightarrow p(7n+5) \equiv 0 \pmod 7$

$X_r \rightarrow p(11n+6) \equiv 0 \pmod{11}$

rank of ptn =

largest part

− number of parts