Relations among Ramanujan-type Congruences

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Goals of this talk

Demonstrate how to make Ramanujan-type congruences accessible to tools from modular representation theory.

Showcase case the power of this connection in the case of integral weights.

Illustrate its potential in the case of half-integral weights through computer calculations.

Discuss the limits of this approach.

Related (forthcoming) papers

Relations among Ramanujan-type congruences I–III.

Integral weight case Discussed essentially exhaustively in part I.

Half-integral weight case Discussed in part II, but the concept seems to be stronger than what I can extract from it now. Hopefully released in early 2021.

Mock theta series Discussed in part III. Basically all results from part II can be transferred to mock theta series (without redoing the previous proofs, but using another technique). Probably in 2021.

Ramanujan-type congruences

Ramanujan congruences

There are three classical Ramanujan congruences

$p(5n+4) \equiv 0 \pmod{5}$, $p(7n+5) \equiv 0 \pmod{7}$, $p(11n+6) \equiv 0 \pmod{11}$.

The crucial fact is that

$$4 - \frac{1}{24} \equiv 0 \pmod{5}, 5 - \frac{1}{24} \equiv 0 \pmod{7}, 6 - \frac{1}{24} \equiv 0 \pmod{11}.$$

So these are U_ℓ-congruences for $1/\eta$.

From weakly holomorphic to cusp forms

In general, projecting to coefficients congruent modulo M to $\beta \in \mathbb{Q}$, we obtain functions

$$\mathrm{U}_{M,\beta}\,\eta^{-1} = \sum_{n \equiv \beta \pmod{M}} p\big(n + \tfrac{1}{24}\big) e\big(n\tau / M\big).$$

If $M = \ell \not\in \{2,3\}$ and β is not in the same square class as $-\frac{1}{24}$, then this is congruent to a cusp form, enabling us to find Ramanujan-type congruences.

Atkin, Ono, Ahlgren-Ono

Through usual congruences for Hecke operators (i.e., genus 0 or Serre) we obtain congruences

$$\forall n \in \mathbb{Z} : p(\ell q^3 n + \beta + \tfrac{1}{24}) \equiv 0 \pmod{\ell}, \quad \ell \in \{5, 7, 13\};$$
$$\forall n \in \mathbb{Z} : p(\ell q^4 n + \beta + \tfrac{1}{24}) \equiv 0 \pmod{\ell}$$

for infinitely many primes q.

More is true: In the case of Atkin, these live on one square class of β with $q^2 \| \beta$; in the case of Ono, Ahlgren-Ono they live on two square classes with $q^3 \| \beta$.

Treneer

For integral weight cusp forms, we obtain the stronger congruences

$$\forall n \in \mathbb{Z} : c(f; q^2 n + \beta) \equiv 0 \pmod{\ell}$$

for infinitely many primes $q \equiv 1 \pmod{\ell}$. These live on the two square classes with $q \| \beta$.

Similarly for weakly holomorphic modular forms, we find some integer m such that

$$\forall n \in \mathbb{Z} : c(f; \ell^m q^2 n + \beta) \equiv 0 \pmod{\ell}.$$

In all cases, the square classes immediately fall out of the construction via Hecke operators.

Ramanujan-type congruences implied by Hecke congruences

Let f be a modular form of integral weight k and level 1 with ℓ-integral Fourier coefficients, ℓ odd. Consider a prime $p \neq \ell$. Assume that we have the Hecke congruence

$$f\big|_k \mathrm{T}_p \equiv \lambda_p f \pmod{\ell}.$$

If $\lambda_p^2 \equiv 4p^{k-1} \pmod{\ell}$, we have the Ramanujan-type congruence with gaps

$$\forall n \in \mathbb{Z} \setminus p\mathbb{Z} : c(f; p^m n) \equiv 0 \pmod{\ell}$$

for a positive integer m if and only if $m \equiv -1 \pmod{\ell}$.
If $\lambda_p^2 \not\equiv 4p^{k-1} \pmod{\ell}$ and the L-polynomial factors over $\overline{\mathbb{F}}_\ell$ as

$$1 - \lambda_p X + p^{k-1} X^2 \equiv (1 - \alpha_p X)(1 - \beta_p X) \pmod{\ell},$$

then we have that congruence if and only if $\alpha_p^{m+1} \equiv \beta_p^{m+1} \pmod{\ell}$.

Upshot

There are plenty of Ramanujan-type congruences implied by Hecke congruences.

What is the precise relation between Ramanujan-type and Hecke congruences?

Square classes

Radu's result

Radu proved through a delicate calculation for all primes ℓ that

$$\forall n \in \mathbb{Z} : p(Mn + \beta) \equiv 0 \pmod{\ell}$$

implies that

$$\forall u \in \mathbb{Z}, \gcd(u, M) = 1 \forall n \in \mathbb{Z} : p(Mn + \beta') \equiv 0 \pmod{\ell}, \\ 24\beta' - 1 \equiv u^2(24\beta - 1) \pmod{M}.$$

Congruences on square classes

The analogue is true for all weakly holomorphic modular forms on $\Gamma(N) \cap \Gamma_0(N')$ for any finite character of $\Gamma_0(N')$:

$$\forall n \in \mathbb{Z} : c(f; Mn + \beta) \equiv 0 \pmod{\ell}$$

implies that

$$\forall u \in \mathbb{Z}, \gcd(u, M) = 1, u \equiv 1 \pmod{N}$$
$$\forall n \in \mathbb{Z} : c(f; Mn + u^2\beta) \equiv 0 \pmod{\ell}.$$

The action of split the Cartan on congruences

The proof is much more conceptual and shows that the phenomenon arises from the action of the split Cartan of $\mathrm{SL}_2(\mathbb{F}_q)$.

For example, the framework is strong enough to show that for a Siegel modular form of level N, $\ell \notin \{2,3\}$, $\beta \in \mathrm{Mat}^{\mathrm{t}}_n(\mathbb{Q})$, and M a lattice in $\mathrm{Mat}^{\mathrm{t}}_n(\mathbb{Q})$:

$$\forall n \in M : c(f; n+\beta) \equiv 0 \pmod{\ell}$$

implies that

$$\forall u \in \mathrm{Mat}_n(\mathbb{Z}), \text{“}\gcd(u, M) = 1\text{”}, u \equiv 1 \pmod{N}$$
$$\forall n \in L : c(f; n + {}^{\mathrm{t}}u\beta u) \equiv 0 \pmod{\ell}.$$

Similarly for Fourier-Jacobi expansions.

Coupling of square classes

The coupling of two square classes as exhibited by the congruences of Ono, Ahlgren-Ono, Treneer is not explained by the Cartan subgroup. Atkin's congruences show it is not true in general.

It is time to specialize on the case of integral weights.

Results for integral weights

Setup

Fix an odd prime ℓ and let f be a modular form of integral weight k for a Dirichlet character χ modulo N with ℓ-integral Fourier coefficients $c(f; n)$. Assume that f satisfies the Ramanujan-type congruence

$$\forall n \in \mathbb{Z} : c(f; Mn + \beta) \equiv 0 \pmod{\ell}$$

for some positive integer M and some integer β.

Relation to Hecke congruences

Consider a prime $p | M$ that does not divide ℓN, and factor M as $M_p M_p^{\#}$ with a p-power M_p and $M_p^{\#}$ co-prime to p. Then

$$f_p := \sum_{n \equiv \beta \pmod{M_p^{\#}}} c(f;n) e(n\tau)$$

is the sum modulo ℓ of T_p-eigenforms $f_{p,\lambda}$, where T_p is the p-th classical Hecke operator. Each $f_{p,\lambda}$ satisfies the Ramanujan-type congruence satisfied by f, and these congruences are implied by the Hecke congruences

$$f_{p,\lambda}\big|_{k,\chi} \mathrm{T}_p \equiv \lambda f_{p,\lambda} \pmod{\ell}.$$

Structure of Ramanujan-type congruences

We have

$$\forall n \in \mathbb{Z} : c(f; M'n + \beta) \equiv 0 \pmod{\ell},$$

where $M' = \mathrm{gcd}(M, M_{\mathrm{sf}} N \beta)$ and M_{sf} is the largest square-free divisor of M.

At $2 | M$ this is stronger than what follows from the square-class result.

Structure of Ramanujan-type congruences

Consider a prime $p | M$ that does not divide ℓN, and factor M as $M_p M_p^{\#}$ with a p-power M_p and $M_p^{\#}$ co-prime to p. Then we have the Ramanujan-type congruence with gap

$$\forall n \in \mathbb{Z} \setminus p\mathbb{Z} : c(f; (M/p)n + M_p\beta') \equiv 0 \pmod{\ell},$$
$$M_p\beta' \equiv \beta \pmod{M_p^{\#}}.$$

Structure of Ramanujan-type congruences

Consider a prime $p | M$ that does not divide ℓN, factor M as $M_p M_p^{\#}$ with a p-power M_p and $M_p^{\#}$ co-prime to p. If $M_p | \beta$, or if $p^2 \nmid M$ and $\ell \neq 2$, then we have the Ramanujan-type congruence

$$\forall n \in \mathbb{Z} : c(f; M_p^{\#} n + \beta) \equiv 0 \pmod{\ell}.$$

In particular, U_p-congruence as opposed to U_ℓ do not occur, and Treneer's congruences are the strongest possible. This also settles the integral weight analogue of recent work with S. Ahlgren and O. Beckwith on the partition function.

The theory of ℓ-kernels

Abstract spaces of modular forms

Let V be a finite dimensional right-module for $\mathbb{C}[\mathrm{SL}_2(\mathbb{Z})]$ such that $\ker_{\mathrm{SL}_2}(V) \subseteq \mathrm{SL}_2(\mathbb{Z})$ has finite index. An abstract space of weakly holomorphic modular forms is a pair of such a module V and a homomorphism of $\mathbb{C}[\mathrm{SL}_2(\mathbb{Z})]$-modules $\phi : V \to \mathrm{M}^!_k(\ker_{\mathrm{SL}_2}(V))$ for some $k \in \mathbb{Z}$.

One can think of this as the dual to vector-valued modular forms, made to work well with representation theory.

The easiest case is $(\mathbb{C}f, \mathrm{id})$ for a modular form f.

The ℓ-kernel

The ℓ-kernel $\ker_{\mathrm{FE}\ell}(V, \phi)$ is the submodule of all $v \in V$ that map to weakly holomorphic modular forms with ℓ-integral Fourier expansions.

Theorem: If V factors through a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$, the ℓ-kernel is a representation for $\Gamma_0(\ell^m)$ for a some specific m (over for instance $\mathscr{O}_{\overline{\mathbb{Q}},\ell}$).

Hecke operators

We have vector-valued Hecke operators

$$\mathrm{T}_M V := V \otimes_{\mathbb{C}[\mathrm{SL}_2(\mathbb{Z})]} \mathbb{C}[\{\gamma \in \mathrm{Mat}_2(\mathbb{Z}) : \det(\gamma) = M\}],$$
$$(\mathrm{T}_M \phi)(v \otimes \gamma) := \phi(v)|_k \gamma.$$

There is an inclusion $\mathrm{Ind}_{\Gamma_0(M)}^{\mathrm{SL}_2(\mathbb{Z})} \psi_M^* V \hookrightarrow \mathrm{T}_M V$ for some pullback ψ_M^*. This allows us later to work with the principal block in the modular representation theory of $\mathrm{SL}_2(\mathbb{F}_q)$.

Key method

Ramanujan-type congruences for $f = \phi(v)$, $v \in V$ correspond to specific vectors $\mathrm{T}_M(v, \beta) \in \mathrm{T}_M V$.

Which representation is generated by these $\mathrm{T}_M(v, \beta)$?
Which $\mathrm{T}_M(v, \beta')$ does it contain?

The coefficient ring is $\mathscr{O}_{\overline{\mathbb{Q}},\ell}$, so we face modular representation theory.

An example calculation

Setting

We go back to the case of half-integral weight, where ℓ-kernels can be also defined and are also representations for a large enough group.

What are the relations among Ramanujan-type congruences modulo 13 of a cusp form of weight $\frac{13^2}{2} - 1$?

This can be implemented on a computer.