A new approach to Dyson's rank conjectures

Frank Garvan
url: qseries.org/fgarvan

University of Florida

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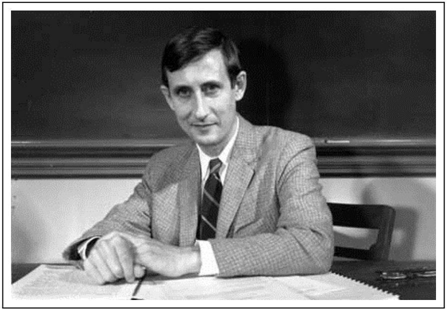
ABSTRACT

- Dyson's rank
- Ramanujan's partition congruences
- Atkin and Swinnerton-Dyer's proof (1954)
- Approach via weak harmonic maass forms (2016)
- Approach via Hecke-Rogers series
- Ramanujan and Dyson's rank in LNB

IN MEMORY OF FREEMAN DYSON (1923 – 2020)



Freeman Dyson (left) answers questions from the audience after the lecture "Heretical thoughts about science and society" organized by the Dynasty Foundation (March 23, 2009, Lebedev Physical Institute (FIAN), Moscow, Russian Federation).

RAMANUJAN (1887 – 1920)

A *partition of n* is a representation of n as a sum of nonincreasing positive integers. Let $p(n)$ denote the number of partitions of n.

n=1	1	p(1)=1
n=2	2	p(2)=2
	1+1	
n=3	3	p(3)=3
	2+1	
	1+1+1	
n=4	4	p(4)=5
	3+1	
	2+2	
	2+1+1	
	1+1+1+1	

$$\vdots$$

Figure: LEONHARD EULER (1707 — 1783)

EULER: For $|q| < 1$

$$P(q) := \sum_{n=0}^{\infty} p(n) q^n = \prod_{m=1}^{\infty} \frac{1}{1-q^m}$$
$$= \frac{1}{\sum_{k=-\infty}^{\infty} (-1)^k q^{k(3k-1)/2}} = \frac{1}{1 - q - q^2 + q^5 + q^7 - \cdots}$$
$$= 1 + \sum_{k=1}^{\infty} \frac{q^{k^2}}{(1-q)^2(1-q^2)^2 \cdots (1-q^k)^2}$$

Let $q = e^{2\pi i \tau}$ with $\mathrm{Im}\tau > 0$, then

$$\eta(\tau) := e^{\pi i/12} \prod_{m=1}^{\infty} (1 - e^{2\pi i \tau m}) = q^{1/24} \prod_{m=1}^{\infty} (1 - q^m)$$

so that $\quad P(q) = \dfrac{q^{1/24}}{\eta(\tau)}.$

Ramanujan's Partition Congruences (1919)

$$p(5n+4) \equiv 0 \pmod{5}$$

$$p(7n+5) \equiv 0 \pmod{7}$$

$$p(11n+6) \equiv 0 \pmod{11}$$

F. J. Dyson, *Some guesses in the theory of partitions*, Eureka (Cambridge) **8** (1944), 10–15.

analytically without much difficulty, using identities like (3); in fact, there are at least four different proofs of (4) and (5).

It would be satisfying to have a direct proof of (4). By this I mean, that although we can prove (in four ways) that the partitions of $5n + 4$ can be divided into five equally numerous subclasses, it is unsatisfactory to receive from the proofs no concrete idea of how the division is to be made. We require a proof which will not appeal to generating functions, but will demonstrate by cross-examination of the partitions themselves the existence of five exclusive, exhaustive and equally numerous subclasses. In what follows I shall not give such a proof, but I shall take the first step towards it, as will appear.

Dyson (1944) defined the *RANK* of a partition as the largest part minus the number of parts.

Example. The rank of $14 + 8 + 2 + 2 + 1$ is $14 - 5 = 9$.

Let $N(m, n)$ denote the number of partitions of n with rank m.

Let $N(m, t, n)$ denote the number of partitions of n with rank $\equiv m \pmod{t}$.

q-NOTATION

$$(a)_n = (a;q)_n = (1-a)(1-aq)(1-aq^2)\cdots(1-aq^{n-1})$$

$$(a)_\infty = (a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k),$$
$$(a)_n = (a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}$$

RANK GENERATING FUNCTION

$$\begin{aligned} R(z,q) &= \sum_{n=0}^{\infty}\sum_{m} N(m,n) z^m q^n \\ &= 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(zq;q)_n (z^{-1}q;q)_n} \\ &= \frac{1}{(q;q)_\infty}\left(1 + \sum_{n=1}^{\infty} \frac{(-1)^n (1+q^n)(1-z)(1-z^{-1})}{(1-zq^n)(1-z^{-1}q^n)}\, q^{\frac{1}{2}n(3n+1)}\right). \end{aligned}$$

DYSON'S RANK CONJECTURES

$$N(k,5,5n+4) = \frac{p(5n+4)}{5} \quad 0 \le k \le 4$$

$$N(k,7,7n+5) = \frac{p(7n+5)}{7} \quad 0 \le k \le 6$$

Atkin and Swinnerton-Dyer (1954)

> The conjecture which I am making is
>
> (8) $N(0, 5, 5n+4) = N(1, 5, 5n+4) = N(2, 5, 5n+4) = N(3, 5, 5n+4) = N(4, 5, 5n+4)$;
>
> or, in words, the partitions of $5n+4$ are divided into five equally numerous classes according to the five possible values of the least positive residue of their ranks modulo 5. In the same way we have
>
> (9) $N(0, 7, 7n+5) = N(1, 7, 7n+5) = \dots = N(6, 7, 7n+5)$.
>
> The truth of (4) and (5) would follow at once, if (8) and (9) could be proved. But the corresponding conjecture with modulus 11 is definitely false.

EXAMPLE

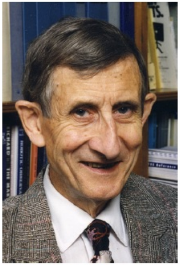
Partitions	Rank	Rank modulo 5
4	3	3
3+1	1	1
2+2	0	0
2+1+1	-1	4
1+1+1+1	-3	2

EQUIVALENT FORM OF DYSON'S RANK CONJECTURES

Let $\zeta_p = \exp(2\pi i/p)$. Then

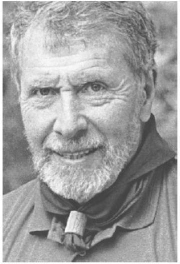
$$R(\zeta_p, q) = \sum_{n\geq 0} \sum_{m} N(m,n)\zeta_p^m q^n$$
$$= \sum_{n\geq 0} \left(\sum_{k=0}^{p-1} N(k,p,n)\zeta_p^k \right) q^n.$$

$$DRC.5 \quad \Longleftrightarrow \quad \text{Coeff of } q^{5n+4} \text{ in } R(\zeta_5, q) = 0$$
$$DRC.7 \quad \Longleftrightarrow \quad \text{Coeff of } q^{7n+5} \text{ in } R(\zeta_7, q) = 0$$



Freeman Dyson

Frank Garvan



Oliver Atkin

H.P.F. Swinnerton-Dyer

Dean Hickerson Eric Mortenson

- Atkin and Swinnerton-Dyer (1954) (see also RLN III by Andrews and Berndt) Identities for theta-functions and Lambert series using elliptic function techniques – Also need to prove the complete 5-dissection etc
- Hickerson and Mortenson (2017) Appell-Lerch series
- G. (2016 – 2018) Theory of harmonic Maass forms - Does NOT need the complete 5-dissection
- G. (2020) Hecke-Rogers series (and q-series and theta-functions) Does NOT need the complete 5-dissection

HECKE-ROGERS SERIES has the form

$$\sum_{(n,m)\in D} (\pm 1)^{f(n,m)} q^{Q(n,m)+L(n,m)}$$

where Q is an indefinite binary quadratic form, L is a linear form and D is a subset of $\mathbb{Z}^2$ for which $Q(n,m) \geq 0$.

EXAMPLE

$$\prod_{n=1}^{\infty}(1-q^n)^2 = \sum_{n=-\infty}^{\infty} \sum_{m=-\lfloor n/2 \rfloor}^{\lfloor n/2 \rfloor} (-1)^{n+m} q^{(n^2-3m^2)/2+(n+m)/2}$$

HECKE (1954) ROGERS (1894) (Биктор Кац) KAC and PETERSON (1980) ANDREWS (1984)

G. (2015), Ji and Zhao (2015) HR-RANK-ID1

$$(zq)_\infty(z^{-1}q)_\infty(q)_\infty \sum_{n=0}^{\infty} \frac{q^{n^2}}{(zq)_n(z^{-1}q)_n} = (zq)_\infty(z^{-1}q)_\infty(q)_\infty R(z,q)$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\left(\sum_{j=0}^{[n/2]}(-1)^{n+j}(z^{n-3j}+z^{3j-n})q^{\frac{1}{2}(n^2-3j^2)+\frac{1}{2}(n-j)} + \sum_{j=1}^{[n/2]}(-1)^{n+j}(z^{n-3j+1}+z^{3j-n-1})q^{\frac{1}{2}(n^2-3j^2)+\frac{1}{2}(n+j)}\right).$$

BRADLEY-THRUSH (2020) HR-RANK-ID2

$$(zq)_\infty (z^{-1}q)_\infty (q)_\infty R(z, q^2) = \sum_{n=0}^{\infty} \sum_{j=-n}^{n} (-1)^j z^j q^{\frac{1}{2}n(3n+1)-j^2} (1-q^{2n+1}).$$

$$\frac{(q)_\infty^3}{(q^2; q^2)_\infty} = \sum_{n=0}^{\infty} \sum_{j=-n}^{n} (-1)^j q^{\frac{1}{2}n(3n+1)-j^2} (1 - q^{2n+1}).$$

HR-RANK-ID3

$$\begin{aligned}(1+z)(z^2q;q)_\infty(z^{-2}q;q)_\infty(q;q)_\infty \sum_{n=0}^{\infty} \frac{q^{n^2}}{(zq)_n(z^{-1}q)_n} \\ = (1+z)(z^2q;q)_\infty(z^{-2}q;q)_\infty(q;q)_\infty R(z,q) \\ = \sum_{n=0}^{\infty} \sum_{j=-\lfloor n/2 \rfloor}^{\lfloor n/2 \rfloor} (-1)^{n+j}(z^{n+1}+z^{-n})q^{\frac{1}{2}(n^2-3j^2)+\frac{1}{2}(n-j)}\end{aligned}$$

Can prove this identity using special case of a Theorem of HICKERSON AND MORTENSON (2014) for their Hecke-Rogers series $f_{1,2,1}$ in terms of Apell-Lerch series.

HR-RANK-ID4

$$
\begin{aligned}
(1+z)(z^2q^2;q^2)_\infty(z^{-2}q^2;q^2)_\infty(q^2;q^2)_\infty \sum_{n=0}^{\infty} \frac{q^{n^2}}{(zq)_n(z^{-1}q)_n} \\
= (1+z)(z^2q^2;q^2)_\infty(z^{-2}q^2;q^2)_\infty(q^2;q^2)_\infty R(z,q) \\
= \sum_{n=0}^{\infty}\sum_{j=0}^{2n}(-1)^n(z^{j+1}+z^{-j})q^{3n^2+2n-\frac{1}{2}j(j+1)} \\
- \sum_{n=1}^{\infty}\sum_{j=0}^{2n-2}(-1)^n(z^{j+1}+z^{-j})q^{3n^2-2n-\frac{1}{2}j(j+1)}
\end{aligned}
$$

THEOREM [BRADLEY-THRUSH (2020)] Let $k \in \mathbb{Z}^+$, $|p|$, $|q|$, $|pq^{-k^2}| < 1$. Then

$$\theta(y;q) \sum_{n=-\infty}^{\infty} \frac{(-1)^n p^{n(n+1)/2} x^n}{1 - yq^k} = \sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} (-1)^{m+n} q^{(n-m)(n-m+1)/2} \theta(q^{-kn}/x;p) y^m,$$

where $\theta(x;q) = (x;q)_\infty (q/x;q)_\infty (q;q)_\infty$.

$k = 2,\ p = q^6,\ x = q^{-2},\ y = z \Longrightarrow$ HR-RANK-ID2

$k = 1,\ p = q^3,\ q \to q^2,\ x = q^{-1}, 1\ y = z^2 \Longrightarrow$ HR-RANK-ID4

Jonathan Bradley-Thrush

JACOBI JTP

$$j(z,q) := (z;q)_\infty(z^{-1}q;q)_\infty(q;q)_\infty = \sum_{n=-\infty}^{\infty}(-1)^n z^n q^{n(n-1)/2}$$

NOTATION

Let a, b be integers where $0 < a < b$.

$$J_{b,a} := j(q^a, q^b) = (q^a;q^b)_\infty(q^{b-a};q^b)_\infty(q^b;q^b)_\infty$$

$$J_b := (q^b;q^b)_\infty$$

Let

$$F(q) = \sum_{n} a(n) q^n \in \mathbb{Z}[[q]]$$

The p-**DISSECTION** of $F(q)$ is

$$F(q) = \sum_{r=0}^{p-1} \sum_{n \equiv r \pmod{p}} a(n) q^n = \sum_{r=0}^{p-1} q^r F_r(q^p)$$

EXAMPLE The 5-DISSECTION of $(\zeta q;q)_\infty(\zeta^{-1}q;q)_\infty(q;q)_\infty$, where $\zeta = \zeta = \exp(2\pi i/5)$. Let $z=\zeta$ in JTP

r	$m(m-1)/2$ where $m=5n+r$
0	$25/2n^2 - 5/2n$
1	$25/2n^2 + 5/2n$
2	$25/2n^2 + 15/2n + 1$
3	$25/2n^2 + 25/2n + 3$
-1	$25/2n^2 - 15/2n + 1$

5-DISS-1

$$(\zeta q)_\infty(\zeta^{-1}q)_\infty(q)_\infty = J_{25,10} + q(\zeta^2+\zeta^{-2})J_{25,5}$$

5-DISS-2

$$E(q) := (q)_\infty = J_{25}\left(\frac{J_{25,10}}{J_{25,5}} - q - q^2\frac{J_{25,5}}{J_{25,10}}\right)$$

5-DISS-3

$$\theta_4(q) = \sum_{n=-\infty}^{\infty}(-1)^n q^{n^2} = J_{50,25} - 2qJ_{50,15} + 2q^4J_{50,5}$$

$$F(q) = \sum_n a(n) q^n$$

$$U_{p,m}(F) = \sum_n a(pn+m) q^n,$$

$$U^*_{p,m}(F) = \sum_n a(pn+m) q^{pn+m},$$

$$A_{p,m}(F) = \sum_n a(n) q^{(n-m)/p},$$

$$U_{p,m} = A_{p,m} \circ U^*_{p,m}.$$

SIFT-1

$$U_{5,2}\left(E(q)^2\right) = -J_5^2$$

SIFT-2

$$U_{5,3}\left(\theta_4(q)\,E(q)\right) = 2\,\frac{J_5 J_{5,1} J_{10,3}}{J_{5,2}}$$

$$U_{5,4}\left(\theta_4(q)\,E(q)\right) = 2\,\frac{J_5 J_{5,2} J_{10,1}}{J_{5,1}}$$

RANK 5-DISS-1

$$U_{5,2}\left((\zeta q)_\infty(\zeta^{-1}q)_\infty(q)_\infty\, R(\zeta,q)\right) = -J_5^2$$

PROOF

$$\frac{1}{2}(n^2-3j^2)+\frac{1}{2}(n-j)\equiv 2 \pmod 5$$

if and only if $n\equiv 2 \pmod 5$ and $j\equiv 4 \pmod 5$, in which case $n-3j\equiv 0 \pmod 5$

$$\frac{1}{2}(n^2-3j^2)+\frac{1}{2}(n+j)\equiv 2 \pmod 5$$

if and only if $n\equiv 2 \pmod 5$ and $j\equiv 1 \pmod 5$, in which case $n-3j+1\equiv 0 \pmod 5$.

$$U_{5,2}\left((\zeta q)_\infty(\zeta^{-1} q)_\infty(q)_\infty\, R(\zeta, q)\right)$$
$$= U_{5,2}\left(\sum_{n=0}^{\infty}\sum_{j=-\lfloor n/2\rfloor}^{\lfloor n/2\rfloor}(-1)^{n+j} q^{\frac{1}{2}(n^2-3j^2)+\frac{1}{2}(n-j)}\right)$$
$$= U_{5,2}\left(E(q)^2\right) = -J^2.$$

□

RANK 5-DISS-2ab

$$U_{5,3}\left((\zeta q)_\infty(\zeta^{-1}q)_\infty(q)_\infty\, R(\zeta, q^2)\right) = (\zeta^2+\zeta^3)\,\frac{J_5 J_{5,1} J_{10,3}}{J_{5,2}}$$

$$U_{5,4}\left((\zeta q)_\infty(\zeta^{-1}q)_\infty(q)_\infty\, R(\zeta, q^2)\right) = (\zeta+\zeta^4)\,\frac{J_5 J_{5,2} J_{10,1}}{J_{5,1}}$$

PROOF SKETCH

$$U_{5,3}\left((\zeta q)_\infty(\zeta^{-1} q)_\infty(q)_\infty\, R(\zeta, q^2)\right) = \frac{1}{2}(\zeta^2+\zeta^3)\,U_{5,3}((q)_\infty^3\, R(1, q^2))$$

$$= \frac{1}{2}(\zeta^2+\zeta^3)\,U_{5,3}(\theta_4(q)\, E(q)) = (\zeta^2+\zeta^3)\,\frac{J_5 J_{5,2} J_{10,1}}{J_{5,1}}$$

Similarly . . . □

$$DRC.5 \iff U_{5,3}(R(\zeta, q^2)) = 0$$

$$R(\zeta, q^2) = \sum_{k=0}^{4} q^k R_k(q^5).$$

EQ-1

$$(\zeta^2+\zeta^3)J_{10,2}\,R_2(q)+J_{10,4}\,R_4(q)=-J_{10}^2$$

EQ-2

$$(\zeta^2+\zeta^3)J_{5,1}\,R_2(q)+J_{5,2}\,R_3(q)=(\zeta^2+\zeta^3)\frac{J_{5,1}J_5J_{10,3}}{J_{5,2}}$$

EQ-3

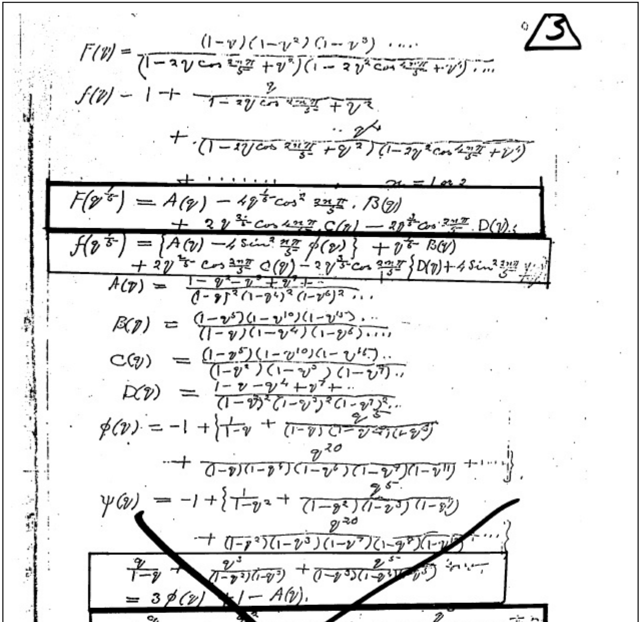
$$(\zeta^2+\zeta^3)J_{5,1}\,R_3(q)+J_{5,2}\,R_4(q)=(\zeta+\zeta^4)\frac{J_{5,2}J_5J_{10,1}}{J_{5,1}}$$

$$
\begin{aligned}
R_3(q) &= \frac{1}{D(q)} \big(J_5\, J_{5,1}^3\, J_{10,2}\, J_{10,3}\, J_{10,4} - 2\, J_5\, J_{5,1}^2\, J_{5,2}\, J_{10,1}\, J_{10,4}^2 \\
&\quad + J_5\, J_{5,2}^3\, J_{10,1}\, J_{10,2}\, J_{10,4} + J_{10}^2\, J_{5,1}^3\, J_{5,2}\, J_{10,4} \\
&\quad - J_{10}^2\, J_{5,1}\, J_{5,2}^3\, J_{10,2} - J_{5,2} \big(J_5\, J_{5,1}^2\, J_{10,1}\, J_{10,4}^2 \\
&\quad + J_5\, J_{5,1}\, J_{5,2}\, J_{10,2}^2\, J_{10,3} - J_5\, J_{5,2}^2\, J_{10,1}\, J_{10,2}\, J_{10,4} \\
&\qquad - J_{10}^2\, J_{5,1}^3\, J_{10,4} \big) \big(\zeta^2 + \zeta^3 \big) \big) \\
&= 0,
\end{aligned}
$$

using

$$
\frac{J_{10,1}\, J_{10,4}}{J_{10}^2} = \frac{J_{5,1}}{J_5}, \qquad \frac{J_{10,2}\, J_{10,3}}{J_{10}^2} = \frac{J_{5,3}}{J_5}.
$$

$$
R_2(q) = \frac{J_{10}^2}{J_{10,2}}, \quad R_4(q) = -(1 + \zeta^2 + \zeta^3)\, \frac{J_{10}^2}{J_{10,4}}.
$$



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$$F(q) = \frac{(1-q)(1-q^2)(1-q^3)\cdots}{(1-2q\cos\frac{2n\pi}{5}+q^2)(1-2q^2\cos\frac{2n\pi}{5}+q^4)\cdots}$$

$$f(q) = 1 + \frac{q}{1-2q\cos\frac{2n\pi}{5}+q^2} + \frac{q^4}{(1-2q\cos\frac{2n\pi}{5}+q^2)(1-2q^2\cos\frac{2n\pi}{5}+q^4)} + \cdots \qquad n = 1 \text{ or } 2$$

$$F(q^{\frac{1}{5}}) = A(q) - 4q^{\frac{1}{5}}\cos^2\frac{2n\pi}{5}\cdot B(q) + 2q^{\frac{2}{5}}\cos\frac{4n\pi}{5}\cdot C(q) - 2q^{\frac{3}{5}}\cos\frac{2n\pi}{5}\cdot D(q),$$

$$f(q^{\frac{1}{5}}) = \left\{A(q) - 4\sin^2\frac{n\pi}{5}\,\phi(q)\right\} + q^{\frac{1}{5}}B(q) + 2q^{\frac{2}{5}}\cos\frac{2n\pi}{5}\,C(q) - 2q^{\frac{3}{5}}\cos\frac{2n\pi}{5}\left\{D(q) + 4\sin^2\frac{2n\pi}{5}\,\frac{\psi(q)}{q}\right\}$$

$$A(q) = \frac{1-q^2-q^3+q^7+\cdots}{(1-q)^2(1-q^4)^2(1-q^6)^2\cdots}$$

$$B(q) = \frac{(1-q^5)(1-q^{10})(1-q^{15})\cdots}{(1-q)(1-q^4)(1-q^6)\cdots}$$

$$C(q) = \frac{(1-q^5)(1-q^{10})(1-q^{15})\cdots}{(1-q^2)(1-q^3)(1-q^7)\cdots}$$

$$D(q) = \frac{1-q-q^4+q^7+\cdots}{(1-q^2)^2(1-q^3)^2(1-q^7)^2\cdots}$$

$$\phi(q) = -1 + \left\{\frac{1}{1-q} + \frac{q^5}{(1-q)(1-q^4)(1-q^6)} + \frac{q^{20}}{(1-q)(1-q^4)(1-q^6)(1-q^9)(1-q^{11})} + \cdots\right\}.$$

$$\psi(q) = -1 + \left\{\frac{1}{1-q^2} + \frac{q^5}{(1-q^2)(1-q^3)(1-q^7)} + \frac{q^{20}}{(1-q^2)(1-q^3)(1-q^7)(1-q^8)(1-q^{12})} + \cdots\right\}$$

$$\frac{q}{1-q} + \frac{q^3}{(1-q^2)(1-q^3)} + \frac{q^5}{(1-q^3)(1-q^4)(1-q^5)} + \cdots = 3\phi(q) + 1 - A(q).$$

RAMA RANK 5-DISS LNB p.20

$$R(\zeta,q) = A(q^5) + (\zeta+\zeta^{-1}-2)\,\phi(q^5) + q\,B(q^5) + (\zeta+\zeta^{-1})\,q^2\,C(q^5) - (\zeta+\zeta^{-1})\,q^3\left\{D(q^5) - (\zeta^2+\zeta^{-2}-2)\frac{\psi(q^5)}{q^5}\right\},$$

where

$$A(q) = \frac{(q^2,q^3,q^5;q^5)_\infty}{(q,q^4;q^5)_\infty^2},\ B(q) = \frac{(q^5;q^5)_\infty}{(q,q^4;q^5)_\infty},\ C(q) = \frac{(q^5;q^5)_\infty}{(q^2,q^3;q^5)_\infty},\ D(q) = \frac{(q,q^4,q^5;q^5)_\infty}{(q^2,q^3;q^5)_\infty^2},$$

and

$$\phi(q) = -1 + \sum_{n=0}^{\infty}\frac{q^{5n^2}}{(q;q^5)_{n+1}(q^4;q^5)_n},\quad \psi(q) = -1 + \sum_{n=0}^{\infty}\frac{q^{5n^2}}{(q^2;q^5)_{n+1}(q^3;q^5)_n}.$$

GENERALIZED RAMANUJAN RANK IDENTITY

G.(2019)

$$q^{-\frac{1}{24}}R(\zeta_p,q) - \left(\frac{12}{p}\right)\sum_{a=1}^{\frac{1}{2}(p-1)}(-1)^a\left(\zeta_p^{3a+\frac{1}{2}(p+1)} + \zeta_p^{-3a-\frac{1}{2}(p+1)} - \zeta_p^{3a+\frac{1}{2}(p-1)} - \zeta_p^{-3a-\frac{1}{2}(p-1)}\right) q^{\frac{a}{2}(p-3a)-\frac{p^2}{24}}\,\Phi_{p,a}(q^p)$$

$$= \frac{1}{\eta(p^2z)}(\text{Weight 1 W.H.MF on } \Gamma_0(p^2)\cap\Gamma_1(p))$$

where

$$\Phi_{p,a}(q) := \begin{cases} \displaystyle\sum_{n=0}^{\infty}\frac{q^{pn^2}}{(q^a;q^p)_{n+1}(q^{p-a};q^p)_n}, & \text{if } 0 < 6a < p, \\ \displaystyle -1 + \sum_{n=0}^{\infty}\frac{q^{pn^2}}{(q^a;q^p)_{n+1}(q^{p-a};q^p)_n}, & \text{if } p < 6a < 3p. \end{cases}$$

$$R_1(q) = \frac{J^2}{J_{5,1}}, \quad R_2(q) = (\zeta + \zeta^{-1})\,\frac{J^2}{J_{5,2}} \quad R_4(q) = 0.$$

It suffices to show that

$$R_0(q) = \frac{{J_5}^2 J_{5,2}}{{J_{5,1}}^2} + \left(\zeta^4 + \zeta - 2\right)\,\phi(q),$$

$$R_3(q) = -\left(\zeta^4 + \zeta\right)\,\frac{J_{5,1}{J_5}^2}{{J_{5,2}}^2} + \frac{1}{q}\,\left(2\,\zeta^3 + 2\,\zeta^2 + 1\right)\,\psi(q).$$

We need HR-RANK-ID3

$$\begin{aligned}
&(1+z)(z^2q;q)_\infty(z^{-2}q;q)_\infty(q;q)_\infty\sum_{n=0}^{\infty}\frac{q^{n^2}}{(zq)_n(z^{-1}q)_n}\\
&\quad=(1+z)(z^2q;q)_\infty(z^{-2}q;q)_\infty(q;q)_\infty R(z,q)\\
&\quad=\sum_{n=0}^{\infty}\sum_{j=-\lfloor n/2\rfloor}^{\lfloor n/2\rfloor}(-1)^{n+j}(z^{n+1}+z^{-n})q^{\frac{1}{2}(n^2-3j^2)+\frac{1}{2}(n-j)}
\end{aligned}$$

Define R-TWIDDLE

$$\widetilde{R}(z,q) = (1+z)(z^2q;q)_\infty(z^{-2}q;q)_\infty(q;q)_\infty R(z,q).$$

LEMMA

$$U_{5,0}\left(\widetilde{R}(\zeta,q)\right) = (1+\zeta)\,\frac{J_5^2\,J_{5,2}^2}{J_{5,1}^2} - (2+2\zeta+\zeta^3)\,\left(\widetilde{R}(q,q^5)-J_{5,2}\right),$$

$$U_{5,4}\left(\widetilde{R}(\zeta,q)\right) = (\zeta^2+\zeta^4)\,\frac{J_5^2\,J_{5,1}^2}{J_{5,2}^2} - \frac{1}{q}\,(2+2\zeta+\zeta^3)\,\left(\widetilde{R}(q^2,q^5)-J_{5,1}\right).$$

$$\frac{1}{2}(n^2-3j^2)+\frac{1}{2}(n-j)\equiv 0 \pmod{5}$$
$$\iff (n,j)\equiv(0,0),(0,3),(1,4),(3,4),(4,0),(4,3) \pmod{5}$$

.

$\zeta^{n+1}+\zeta^{-n}$	n	j
$1+\zeta$	0	0
$1+\zeta$	0	3
$-\zeta^3-\zeta-1$	1	4
$-\zeta^3-\zeta-1$	3	4
$1+\zeta$	4	0
$1+\zeta$	4	3

$$\mathcal{S}_1=\{(0,0),\,(0,3),\,(1,4),\,(3,4),\,(4,0),\,(4,3)\},\quad \mathcal{S}_2=\{(1,4),\,(3,4)\},$$

$$U_{5,0}^{*}\left(\widetilde{R}(\zeta, q)\right) = (1+\zeta) \sum_{\substack{n=0 \\ (n,j)\in\mathcal{S}_1}}^{\infty} \sum_{\substack{j=-\lfloor n/2\rfloor \\ \pmod 5}}^{\lfloor n/2\rfloor} (-1)^{n+j} q^{\frac{1}{2}(n^2-3j^2)+\frac{1}{2}(n-j)}$$

$$- (2+2\zeta+\zeta^3) \sum_{\substack{n=0 \\ (n,j)\in\mathcal{S}_2}}^{\infty} \sum_{\substack{j=-\lfloor n/2\rfloor \\ \pmod 5}}^{\lfloor n/2\rfloor} (-1)^{n+j} q^{\frac{1}{2}(n^2-3j^2)+\frac{1}{2}(n-j)}$$

$$\widetilde{R}(q, q^5) = \sum_{n=0}^{\infty} \sum_{j=-\lfloor n/2\rfloor}^{\lfloor n/2\rfloor} (-1)^{n+j} \left(q^{n+1} + q^{-n}\right) q^{\frac{5}{2}(n^2-3j^2)+\frac{5}{2}(n-j)}.$$

$$V(n,j) = \frac{1}{2}(n^2 - 3j^2) + \frac{1}{2}(n - j)$$

$$\frac{1}{5}V(5n+1, -5j-1) = 5\,V(n,j) - n,$$
$$\frac{1}{5}V(5n+3, -5j-1) = 5\,V(n,j) + n + 1.$$

$$-\lfloor \tfrac{1}{2}(5n+1) \rfloor \leq -5j-1 \leq \lfloor \tfrac{1}{2}(5n+1) \rfloor \Leftrightarrow \begin{cases} -m \leq j \leq m-1 & n=2m, \\ -m \leq j \leq m & n=2m+1 \end{cases}$$

and

$$-\lfloor \tfrac{1}{2}(5n+3) \rfloor \leq -5j-1 \leq \lfloor \tfrac{1}{2}(5n+3) \rfloor \Leftrightarrow \begin{cases} -m \leq j \leq m & n=2m, \\ -m-1 \leq j \leq m & n=2m+1 \end{cases}$$

$$A_{5,0}\left(\sum_{\substack{n=0 \\ (n,j)\in\mathcal{S}_2 \pmod 5}}^{\infty}\sum_{j=-\lfloor n/2\rfloor}^{\lfloor n/2\rfloor}(-1)^{n+j}q^{\frac{1}{2}(n^2-3j^2)+\frac{1}{2}(n-j)}\right)$$

$$=\sum_{n=0}^{\infty}\sum_{j=-\lfloor n/2\rfloor}^{\lfloor n/2\rfloor}(-1)^{n+j}\left(q^{n+1}+q^{-n}\right)q^{\frac{5}{2}(n^2-3j^2)+\frac{5}{2}(n-j)}$$

$$-\sum_{m=-\infty}^{\infty}(-1)^m q^{m(5m+1)/2}$$

$$=\widetilde{R}(q,q^5)-J_{5,2}$$

$$\sum_{\substack{n=0 \\ (n,j)\in\mathcal{S}_1 \pmod 5}}^{\infty} \sum_{j=-\lfloor n/2\rfloor}^{\lfloor n/2\rfloor} (-1)^{n+j} q^{\frac{1}{2}(n^2-3j^2)+\frac{1}{2}(n-j)}$$

$$= U_{5,0}^{*}\left(\sum_{n=0}^{\infty}\sum_{j=-\lfloor n/2\rfloor}^{\lfloor n/2\rfloor}(-1)^{n+j} q^{\frac{1}{2}(n^2-3j^2)+\frac{1}{2}(n-j)}\right)$$

$$= U_{5,0}^{*}\left((q)_\infty^2\right)$$

$$= \frac{J_{25}^2\, J_{25,10}^2}{J_{25,5}^2},$$

$$U_{5,0}\left(\widetilde{R}(\zeta,q)\right) = (1+\zeta)\,\frac{J^2\,J_{5,2}^2}{J_{5,1}^2} - (2+2\zeta+\zeta^3)\,\left(\widetilde{R}(q,q^5) - J_{5,2}\right)$$

Similarly the result for

$$U_{5,4}\left(\widetilde{R}(\zeta,q)\right) = \cdots$$

$$\begin{aligned}\widetilde{R}(q,q^5) &= (q^2;q^5)_\infty(q^3;q^5)_\infty(q^5;q^5)_\infty\frac{1}{1-q}R(q,q^5)\\ &= J_{5,2}\sum_{n=0}^{\infty}\frac{q^{5n^2}}{(q;q^5)_{n+1}(q^4;q^5)_n}\\ &= J_{5,2}\,(1+\phi(q))\,.\end{aligned}$$

$$\widetilde{R}(\zeta,q) = (1+\zeta)(\zeta^2q;q)_\infty(\zeta^{-2}q;q)_\infty(q;q)_\infty R(\zeta,q)$$

$$\begin{aligned}\widetilde{R}(\zeta,q) = (1+\zeta)\left(J_{25,10}+q(\zeta+\zeta^4)J_{25,5}\right)\Big(R_0(q^5)+\\ q\frac{J_{25}^2}{J_{25,5}}+q^2(\zeta+\zeta^4)\frac{J_{25}^2}{J_{25,10}}+q^3R_3(q^5)\Big)\end{aligned}$$

$$U_{5,0}\left(\widetilde{R}(\zeta,q)\right) = (1+\zeta)\,J_{5,2}\,R_0(q).$$

$$(1+\zeta)\,J_{5,2}\,R_0(q) = (1+\zeta)\frac{J^2 J_{5,2}^2}{J_{5,1}^2} - (2+\zeta+\zeta^3)\,J_{5,2}\,\phi(q),$$

$$R_0(q) = \frac{{J_5}^2 J_{5,2}}{{J_{5,1}}^2} + \left(\zeta^4+\zeta-2\right)\,\phi(q)$$

Similarly

$$R_3(q) = \cdots$$

THANK YOU

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