Quantum q-series identities

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References

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q-series

A q-hypergeometric series (or "q-series") is a series built using the q-Pochhammer symbols

$$(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})$$

For example,

$$\sum_{n\geq 0} \frac{(a;q)_n(b;q)_n z^n}{(q;q)_n(c;q)_n}.$$

Classical q-series identites

As analytic identities, classical q-series identities are identities between functions for $|q| < 1$.

For example,

$$\sum_{n\geq 0} \frac{q^n}{(q;q)_n} = \frac{1}{(q;q)_\infty} (\mathrm{Euler})$$

and

$$\sum_{n\geq 0} \frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty} (\mathrm{Rogers-Ramanujan}).$$

Here

$$(a;q)_\infty = \prod_{k=0}^{\infty} (1 - aq^k).$$

Classical q-series identities

There are hundreds of papers devoted to finding such identities.

Perhaps the most famous are the Andrews-Gordon identities,

$$\sum_{n_{k-1}\geq\cdots\geq n_1\geq 0} \frac{q^{n_{k-1}^2+\cdots+n_1^2+n_{k-i}+\cdots+n_1}}{(q;q)_{n_{k-1}-n_{k-2}}\cdots(q;q)_{n_2-n_1}(q;q)_{n_1}} = \prod_{n\not\equiv 0,\pm i \pmod{2k+1}} \frac{1}{1-q^n}.$$

q-series identities play a role in combinatorics, number theory, statistical mechanics, vertex operator algebras, etc.

q-series are everywhere!

Another type of q-series identity

Let

$$\sigma(q) = \sum_{n\geq 0} \frac{q^{\binom{n+1}{2}}}{(-q;q)_n} = 1 + \sum_{n\geq 0} (-1)^n q^{n+1} (q;q)_n$$

and

$$\sigma^*(q) = 2\sum_{n\geq 1} \frac{(-1)^n q^{n^2}}{(q;q^2)_n} = -2\sum_{n\geq 0} q^{n+1} (q^2;q^2)_n.$$

Note that since

$$(q;q)_n = (1-q)\cdots(1-q^n),$$

the right hand sides are well-defined both when $|q| < 1$ and when $q = e^{2\pi i a/N}$ is a root of unity.

Another type of q-series identity

H. Cohen (1988) showed that if q is any root of unity, then

$$\sigma(q) = -\sigma^*(q^{-1}).$$

For example,

$$\sigma(i) = -2i - 4, \quad \sigma^*(-i) = 2i + 4.$$

Note that $\sigma(q) = -\sigma^*(q^{-1})$ is not true for $|q| < 1$.

Another type of q-series identity

Let $F(q)$ be the Kontsevich-Zagier series

$$F(q) = \sum_{n \geq 0} (q; q)_n$$

and

$$U(q) = \sum_{n \geq 0} (q; q)_n^2 q^{n+1}.$$

Bryson-Ono-Pitman-Rhoades (2012) proved that if q is any root of unity, then

$$F(q^{-1}) = U(q).$$

Again, note that this is not true for $|q| < 1$.

Another type of q-series identity

Let

$$F_k(x,q) = \sum_{n=0}^{k-1} x^{n+1}(xq;q)_n$$

and

$$U_k(x,q) = \sum_{n=0}^{k-1} (-xq;q)_n(-x^{-1}q;q)_n q^{n+1}.$$

Folsom, Ki, Vu and Yang (2016) proved that if q is a kth root of unity and $x \in \mathbb{C}$ then

$$F_k(x,q^{-1}) = x^k U_k(-x,q).$$

Quantum q-series identities

We call such identities quantum q-series identities.

By a *quantum q-series identity* we mean a q-series identity which holds at roots of unity but not for $|q| < 1$.

We write

$$f(q) =_q g(q)$$

if the q-series agree at roots of unity and

$$f(q) =_{q^{-1}} g(q)$$

if (as in the examples above) $f(q) = g(q^{-1})$ for roots of unity q.

Questions

1. Where do such identities come from?
2. How can we find more of them?
3. What are they used for?

In this talk I will focus on the first two questions.

More quantum identities

We will see many more quantum q-series identites, like

$$\sigma^*(q) =_{q^{-1}} -2\sum_{n\geq 0}(q;q)_{2n}q^{2n+1}$$

$$q\sum_{n\geq 0}(q;q)_n q^{n(n+1)/2} =_{q^{-1}} \sum_{n\geq 0}(q;q)_{2n}(q;q^2)_n q^{2n}$$

More quantum identities

$$\sum_{k_t\geq\cdots\geq k_1\geq 1}(q;q)_{k_t-1}^2 q^{k_t}\prod_{i=1}^{t-1} q^{k_i^2}\begin{bmatrix} k_{i+1}+k_i-i+2\sum_{j=1}^{i-1}k_j \\ k_{i+1}-k_i\end{bmatrix}$$

$$=_{q^{-1}}\sum_{k_t\geq\cdots\geq k_1\geq 0}^{\infty}(q;q)_{k_t}\prod_{i=1}^{t-1} q^{k_i(k_i+1)}\begin{bmatrix} k_{i+1} \\ k_i\end{bmatrix}.$$

Here we have used the usual q-binomial coefficient,

$$\begin{bmatrix} n \\ k\end{bmatrix}=\begin{cases}\frac{(q;q)_n}{(q;q)_{n-k}(q;q)_k} & \text{, if } 0\leq k\leq n,\\ 0, & \text{otherwise.}\end{cases}$$

The first observation

While existing proofs use "delicate recursions", quantum q-series identities can be proved using classical q-series transformations.

For example, take Sears' transformation: for $N \in \mathbb{N}$,

$$\sum_{n\geq 0} \frac{(q^{-N};q)_n(b;q)_n(c;q)_n q^n}{(q;q)_n(d;q)_n(e;q)_n} = \frac{(e/c;q)_N c^N}{(e;q)_N} \sum_{n\geq 0} \frac{(q^{-N};q)_n(c;q)_n(d/b;q)_n(bq/e)^n}{(q;q)_n(d;q)_n(cq^{1-N}/e;q)_n}.$$

Setting $N = N-1$, $b = bq^{1+N}$, $c = q$ and letting $d, e \to 0$ we obtain

The first observation

$$\sum_{n\geq 0}(q^{1-N};q)_n(bq^{1+N};q)_n q^{n+1} = q^N \sum_{n\geq 0}(q^{1-N};q)_n q^{2Nn-\binom{n+1}{2}}(-b)^n.$$

Setting $b=-1$ and $q=\zeta_N^a$ and using

$$(-1)^n q^{n(n+1)/2}(q^{-1};q^{-1})_n = (q;q)_n$$

gives the quantum q-series identity of Cohen.

Setting $b=1$ and $q=\zeta_N^a$ gives the quantum q-series identity of Bryson-Ono-Pitman-Rhoades.

The Folsom-Ki-Vu-Yang example can be proved similarly.

Other cases

Other cases of Sears' transformation give new quantum q-series identities.

Take $q = q^2$ and $b = q^{-1}$ above. Then we have

$$\sum_{n\geq 0}(q)_{2n}q^{2n+2} =_{q^{-1}} \sum_{n\geq 0}(q^2;q^2)_n q^n.$$

For example,

$$\mathrm{LHS}(i) = \mathrm{RHS}(-i) = 1 - 2i.$$

Other cases

Some other cases include

$$\sum_{n\geq 0} \frac{(q;q)_n^2 q^n}{(-q;q)_n} =_{q^{-1}} 2q \sum_{n\geq 0} \frac{(q;q)_n}{(-q;q)_{n+1}},$$

$$\sum_{n\geq 0} \frac{(q^2;q^2)_n^2 q^{2n+2}}{(q;q^2)_{n+1}} =_{q^{-1}} \sum_{n\geq 0} \frac{(q^2;q^2)_n}{(q;q^2)_{n+1}},$$

$$\sum_{n\geq 0} (q^2;q^4)_n q^{2n} =_{q^{-1}} q \sum_{n\geq 0} (q;q^2)_n (-1)^n q^n.$$

Here we implicitly assume that the roots of unity are appropriately restricted.

Other classical q-series identities

What about using other classical q-series summations and transformations and identities to find quantum q-series identities?

Take the q-Chu Vandermonde summation,

$$\sum_{n\geq 0} \frac{(q^{-N};q)_n (a;q)_n q^n}{(c;q)_n (q;q)_n} = \frac{(c/a;q)_N a^N}{(c;q)_N}$$

Letting $N = N-1$ and $a = q$ we have

$$\sum_{n\geq 0} \frac{(q^{1-N};q)_n q^n}{(c;q)_n} = \frac{(1-c/q)q^{N-1}}{1-cq^{N-2}}.$$

Other classical q-series identities

Taking q to be an Nth root of unity we obtain evaluations like

$$\sum_{n\geq 0}(q;q)_n q^{n+1} =_q 1$$

and

$$\sum_{n\geq 0}\frac{(q;q)_n q^n}{(-q;q)_n} =_q \frac{2}{1+q}.$$

Similar results follow from summation identities like the q-Pfaff-Saalschutz identity and Jackson's identity.

Other classical q-series identities

Take a quadratic transformation of Jain,

$$\sum_{n\geq 0}\frac{(q^{-N};q^2)_n(q^{1-N};q^2)_n(a;q)_{2n}q^{2n}}{(q^2;q^2)_n(bq;q^2)_n(d;q)_{2n}}$$
$$=\frac{(d/a;q)_N a^N}{(d;q)_n}\sum_{n\geq 0}\frac{(q^{-N};q)_n(a;q)_n(b;q^2)_n(-q/d)^n}{(q;q)_n(b;q)_n(aq^{1-N}/d;q)_n(-1)^n q^{n(n-1)/2}}.$$

The case $a=q, N=N-1, b=d=0$ gives

$$q\sum_{n\geq 0}(q;q)_n q^{n(n+1)/2} =_{q^{-1}} \sum_{n\geq 0}(q;q)_{2n}(q;q^2)_n q^{2n}.$$

Other classical q-series identities

Many other quantum identities can be deduced from Jain's transformation, as well as from transformations of Singh, Watson, etc.

Two of the nicest ones are

$$\sigma(q) =_q \sum_{n\geq 0} \frac{(q;q)_n(1+q^{2n+1})(-1)^n q^{n(3n+1)/2}}{(-q;q)_n}$$

and

$$\sum_{n\geq 0} \frac{(q;q)_n(-1)^n q^{n(n+1)/2}}{(-q;q)_n} =_q 2\sum_{n\geq 0} \frac{(q^2;q^2)_n(-1)^n}{(-q;q^2)_{n+1}}.$$

Summary of Part I

- As analytic identities, classical q-series identities hold for $|q| < 1$.
- There are examples of identities which hold only at roots of unity – quantum q-series identities.
- These have connections to mock theta functions and quantum modular forms, but are also interesting in their own right.
- Quantum identities can be proved using classical q-series identities and transformations.

The second observation

The Bryson-Ono-Pitman-Rhoades identity can be proved using colored Jones polynomials in knot theory!

The colored Jones polynomial $J_N(K) = J_N(K; q)$ is an important knot invariant. It generalizes the classical Jones polynomial (the case $N = 2$).

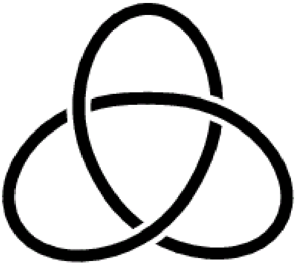
$J_N(K; e^{2\pi i/N})$ appears in the "Volume Conjecture."

If K^* denotes the mirror image of the knot K, then we have the duality

$$J_N(K; q) = J_N(K^*; q^{-1}).$$

The second observation

Consider the trefoil knot:



The second observation

There are actually two of them, a “right-handed” and a ”left-handed” trefoil.

They are mirror images of each other.

The second observation

Let $T_{(2,3)}$ and $T^*_{(2,3)}$ denote the right-handed and left-handed trefoils, respectively.

Formulas of Habiro, Lê and Masbuam give that

$$J_N(T^*_{(2,3)};q) = \sum_{n=0}^{\infty} q^n \,(q^{1-N};q)_n\,(q^{1+N};q)_n$$

and

$$J_N(T_{(2,3)};q) = q^{1-N}\sum_{n=0}^{\infty}(q^{1-N};q)_n q^{-nN}.$$

The second observation

Thus when q is any Nth root of unity we have

$$J_N(T^*_{(2,3)}; q) = q^{-1} U(q)$$

and

$$J_N(T_{(2,3)}; q) = qF(q).$$

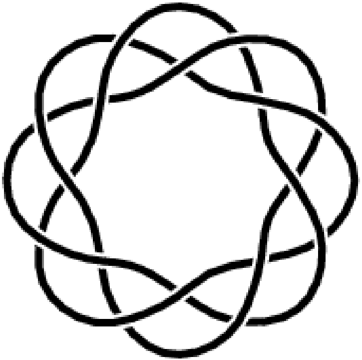
The duality of the colored Jones polynomials then gives

$$F(q) =_{q^{-1}} U(q).$$

Other knots - torus knots

What about other knots?

Consider the torus knots $T_{(s,t)}$.



Other knots - torus knots

The trefoil is a special case of the torus knots $(2, 2t+1)$ for $t \geq 1$.

Hikami showed that the colored Jones polynomial of the right-handed torus knot $T_{(2,2t+1)}$ is

$$J_N(T_{(2,2t+1)}; q) = q^{t(1-N)} \sum_{k_t \geq \cdots \geq k_1 \geq 0}^{\infty} (q^{1-N}; q)_{k_t} q^{-Nk_t} \times \prod_{i=1}^{t-1} q^{k_i(k_i+1-2N)} \begin{bmatrix} k_{i+1} \\ k_i \end{bmatrix}_q .$$

What about the left-handed torus knots $T^*_{(2,2t+1)}$?

Other knots - torus knots

Habiro (2008) defined the cyclotomic expansion of the colored Jones polynomial for a knot K to be

$$J_N(K;q) = \sum_{n=0}^{\infty} C_n(K;q)\,(q^{1+N};q)_n\,(q^{1-N};q)_n$$

and proved that

$$C_n(K;q) \in \mathbb{Z}[q,q^{-1}].$$

The $C_n(K;q)$ are called the cyclotomic coefficients. For the trefoil knot $T^*_{(2,3)}$ we have $C_n = q^n$.

The cyclotomic expansion was known for select families of knots, but not for the torus knots $T^*_{(2,2t+1)}$.

Other knots - torus knots

In the theory of q-hypergeometric series, a Bailey pair relative to a is a pair of sequences $(\alpha_n, \beta_n)_{n\geq 0}$ satisfying

$$\beta_n = \sum_{j=0}^{n} \frac{\alpha_j}{(q;q)_{n-j}(aq;q)_{n+j}},$$

or equivalently,

$$\alpha_n = \frac{1-aq^{2n}}{1-a} \frac{(a;q)_n}{(q;q)_n} (-1)^n q^{n(n-1)/2} \sum_{j=0}^{n} (q^{-n};q)_j (aq^n;q)_j q^j \beta_j.$$

Thus the colored Jones polynomial and its cyclotomic coefficients are essentially a Bailey pair relative to q^2!

Other knots - torus knots

How can we use this?

First, using the "Rosso-Jones formula" we show that

$$(1-q^N)\,J_N(T^*_{(2,2t+1)}) = (-1)^N q^{-t+\frac{1}{2}N+\frac{2t+1}{2}N^2} \times \sum_{k=-N}^{N-1} (-1)^k q^{-\frac{2t+1}{2}k(k+1)+k}.$$

This is the α side of a Bailey pair.

Other knots - torus knots

Next we use results on Bailey pairs and indefinite quadratic forms (L., 2014!) to find the β side.

The (preliminary) result is

$$-q^{t-n}C_{n-1}(T^{*}_{(2,2t+1)};q) = \sum \frac{q^{\sum_{i=1}^{t-1} n_{t+i}^2 + \binom{n_t}{2} - \sum_{i=1}^{t-1} n_i n_{i+1} - \sum_{i=1}^{t-2} n_i}(-1)^{n_t}(1-q^{n_t - \chi(t\geq 2)n_{t-1}})}{(q;q)_{n-n_{2t-1}}(q;q)_{n_{2t-1}-n_{2t-2}}\cdots(q;q)_{n_2-n_1}(q;q)_{n_1}},$$

where the sum is over $n \geq n_{2t-1} \geq \cdots \geq n_1 \geq 0$.

Why is this a (Laurent) polynomial?

Other knots - torus knots

Using further q-series techniques, we reduce the $2t$-fold sum to a t-fold sum.

Theorem (Hikami-L. (2015))

We have

$$C_{n-1}(T^*_{(2,2t+1)};q) =$$

$$q^{n+1-t} \sum_{n+1=k_t \geq k_{t-1} \geq \cdots \geq k_1 \geq 1} \prod_{i=1}^{t-1} q^{k_i^2} \begin{bmatrix} k_{i+1}+k_i-i+2\sum_{j=1}^{i-1} k_j \\ k_{i+1}-k_i \end{bmatrix}.$$

Other knots - torus knots

Corresponding to the torus knots $T_{(2,2t+1)}$ for $t \geq 1$, define the generalized U-function $U_t(q)$ by

$$U_t(q) := q^{-t} \sum_{k_t \geq \cdots \geq k_1 \geq 1} (q;q)_{k_t - 1}^2 q^{k_t} \times \prod_{i=1}^{t-1} q^{k_i^2} \begin{bmatrix} k_{i+1} + k_i - i + 2\sum_{j=1}^{i-1} k_j \\ k_{i+1} - k_i \end{bmatrix}$$

and the generalized Kontsevich-Zagier function by

$$F_t(q) = q^t \sum_{k_t \geq \cdots \geq k_1 \geq 0}^{\infty} (q;q)_{k_t} \prod_{i=1}^{t-1} q^{k_i(k_i+1)} \begin{bmatrix} k_{i+1} \\ k_i \end{bmatrix}.$$

Other knots - torus knots

Theorem (Hikami-L., 2015)

We have $F_t(q) =_{q^{-1}} U_t(q)$.

For example, when $t = 1$ we have

$$\sum_{n\geq 0}(q;q)_n^2 q^n =_{q^{-1}} q\sum_{n\geq 0}(q;q)_n.$$

When $t = 2$ we have

$$\sum_{k_2\geq k_1\geq 0}(q;q)_{k_2} q^{k_1^2+k_1}\begin{bmatrix}k_2\\k_1\end{bmatrix} =_{q^{-1}} \sum_{k_2\geq k_1\geq 1}(q;q)_{k_2-1}^2 q^{k_2+k_1^2}\begin{bmatrix}k_2+k_1-1\\k_2-k_1\end{bmatrix}.$$

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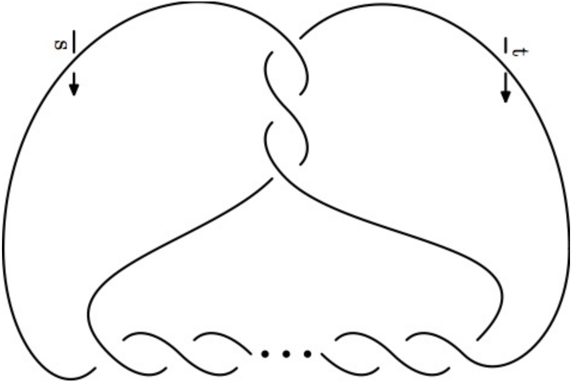
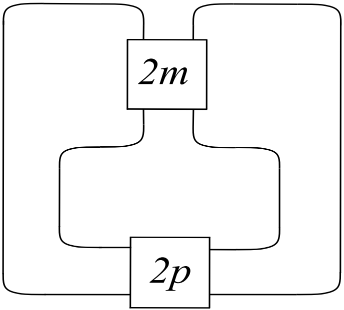
Other knots - twist knots

Consider the double twist knots.



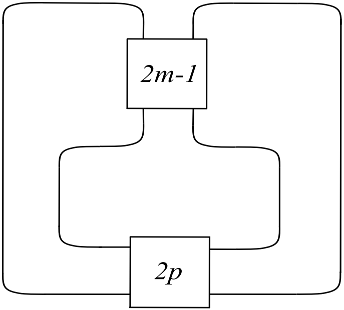
Other knots - twist knots

There are even double twist knots...



Other knots - twist knots

And odd double twist knots.



Other knots - twist knots

• L.-Osburn (2017), q-hypergeometric formulas for the colored Jones polynomials of even twist knots $\Rightarrow$ Two infinite families of quantum q-series identities.

• L.-Osburn (2019), q-hypergeometric formulas for the colored Jones polynomials of odd twist knots $\Rightarrow$ Two infinite families of quantum q-series identities.

Other knots - twist knots

For integers $m, p \geq 1$ define

$$\mathfrak{F}_{m,p}(q)$$

$$= q^{1-p} \sum_{n_{(2m+1)p-1} \geq \cdots \geq n_1 \geq 0} (q;q)_{n_{(2m+1)p-1}} (-1)^{n_{(2m+1)p-1}} q^{-\binom{n_{(2m+1)p-1}+1}{2}}$$

$$\times \prod_{\substack{1 \leq i < j \leq (2m+1)p-1 \\ (2m+1) \nmid i \\ j \not\equiv m \pmod{2m+1}}} q^{\epsilon_{i,j,m} n_i n_j} \prod_{\substack{i=1 \\ i \equiv m, 2m+1 \pmod{2m+1}}}^{(2m+1)p-2} (-1)^{n_i} q^{\binom{n_i+1}{2}}$$

$$\times \prod_{i=1}^{(2m+1)p-2} q^{-n_i n_{i+1} + \gamma_{i,m} n_i} \begin{bmatrix} n_{i+1} \\ n_i \end{bmatrix},$$

Other knots - twist knots

where $\epsilon_{i,j,m}$ and $\gamma_{i,m}$ are defined by

$$\epsilon_{i,j,m} = \begin{cases} 1, & \text{if } j \equiv -i \text{ or } -i-1 \pmod{2m+1}, \\ -1, & \text{if } j \equiv i \text{ or } i-1 \pmod{2m+1}, \\ 0, & \text{otherwise} \end{cases}$$

where $1 \leq i < j \leq (2m+1)p - 1$ with $(2m+1) \nmid i$ and $j \not\equiv m \pmod{2m+1}$ and

$$\gamma_{i,m} = \begin{cases} 1, & \text{if } i \equiv 1, \ldots, m-1 \pmod{2m+1}, \\ -1 & \text{if } i \equiv 0, m+1, \ldots, 2m \pmod{2m+1}, \\ 0 & \text{if } i \equiv m \pmod{2m+1} \end{cases}$$

where $1 \leq i \leq (2m+1)p - 2$.

Other knots - twist knots

Next for integers $m, p \geq 1$ define

$$\mathfrak{U}_{m,p}(q) = q^p \sum_{\substack{n \geq 0 \\ n = n_m \geq n_{m-1} \geq \cdots \geq n_1 \geq 0 \\ n = s_p \geq s_{p-1} \geq \cdots \geq s_1 \geq 0}} \frac{(q;q)_n^2}{(q;q)_{n_1}} q^n \times \prod_{i=1}^{m-1} q^{n_i^2+n_i} \begin{bmatrix} n_{i+1} \\ n_i \end{bmatrix} \prod_{j=1}^{p-1} q^{s_j^2+s_j} \begin{bmatrix} s_{j+1} \\ s_j \end{bmatrix}.$$

Other knots - twist knots

Theorem (L.-Osburn, 2019)

We have $\mathfrak{F}_{m,p}(q) =_{q^{-1}} \mathfrak{U}_{m+1,p}(q)$.

For example, $m = 3$ and $p = 1$ gives

$$\sum_{n_6 \geq n_5 \geq n_4 \geq n_3 \geq n_2 \geq n_1 \geq 0} (q;q)_{n_6} (-1)^{n_3+n_6} q^{\binom{n_3+1}{2} - \binom{n_6+1}{2}} \times q^{n_1(n_5+n_6)+n_2(n_4+n_5)-n_1 n_2 - n_2 n_3 - n_4 n_5 - n_5 n_6} \times q^{n_1+n_2-n_4-n_5} \begin{bmatrix} n_6 \\ n_5 \end{bmatrix} \begin{bmatrix} n_5 \\ n_4 \end{bmatrix} \begin{bmatrix} n_4 \\ n_3 \end{bmatrix} \begin{bmatrix} n_3 \\ n_2 \end{bmatrix} \begin{bmatrix} n_2 \\ n_1 \end{bmatrix}$$

$$=_{q^{-1}} \sum_{\substack{n \geq 0 \\ n = n_4 \geq n_3 \geq n_2 \geq n_1 \geq 0}} \frac{(q;q)_n^2}{(q;q)_{n_1}} q^{n+1+n_3^2+n_3+n_2^2+n_2+n_1^2+n_1} \begin{bmatrix} n_4 \\ n_3 \end{bmatrix} \begin{bmatrix} n_3 \\ n_2 \end{bmatrix} \begin{bmatrix} n_2 \\ n_1 \end{bmatrix}.$$

Other knots - twist knots

- The proof follows from two different formulas for the colored Jones polynomial of the odd double twist knots together with the fact that the (s, t) double twist knot is the mirror image of the $(-s, -t)$ double twist knot.

- The first formula is deduced from work of Takata on "two-bridge" knots.

- The second uses a "skein-theoretic" formula of Walsh + Bailey pairs!

Other knots - twist knots

Specifically, Walsh's work implies that for $(2m-1, 2p)$ twists the colored Jones polynomial is

$$q^{p(1-N^2)} \sum_{n\geq 0} q^n (q^{1+N};q)_n (q^{1-N};q)_n c_{p,n}(q) d_{m,n}(q),$$

where

$$c_{p,n}(q) = (q;q)_n \sum_{k=0}^{n} \frac{(-1)^k q^{\binom{k}{2}+p(k^2+k)}(1-q^{2k+1})}{(q;q)_{n-k}(q;q)_{n+k+1}}$$

and

$$d_{m,n}(q) = (q;q)_n \sum_{k=0}^{n} \frac{q^{mk^2+(m-1)k}(1-q^{2k+1})}{(q;q)_{n-k}(q;q)_{n+k+1}}.$$

Other knots - twist knots

Using Bailey pairs, one can then show that

$$c_{p,n}(q) = \sum_{n=n_p \geq n_{p-1} \geq \cdots \geq n_1 \geq 0} \prod_{j=1}^{p-1} q^{n_j^2+n_j} \begin{bmatrix} n_{j+1} \\ n_j \end{bmatrix}.$$

and

$$d_{m,n}(q) = \sum_{n=n_m \geq n_{m-1} \geq \cdots \geq n_1 \geq 0} \frac{1}{(q;q)_{n_1}} \prod_{j=1}^{m-1} q^{n_j^2+n_j} \begin{bmatrix} n_{j+1} \\ n_j \end{bmatrix}.$$

Summary of Part II

- Quantum q-series identities can be proved using colored Jones polynomials in knot theory.
- We need a formula for the colored Jones polynomial of a knot and another for the colored Jones polynomial of its mirror image.
- Some formulas are known for classes of knots, and some have to be computed.
- Bailey pairs play an important role (and so does work of Takata).

Concluding Remarks

Find more quantum q-series identities. Knots? (Yuasa, Stosic-Wedrich) q-series?

Prove the quantum q-series identities from knot theory using known (classical) q-series identities.

In some cases one can use the "tail" of the colored Jones polynomial to find an identity for $|q| < 1$.

Concluding Remarks

Recall the Kontsevich-Zagier function

$$F(q) = \sum_{n \geq 0} (q;q)_n.$$

The coefficients of the series

$$F(1-q) = 1 + q + 2q^2 + 5q^3 + 15q^4 + 53q^5 + \cdots$$

are called the Fishburn numbers.

They count several important combinatorial objects.

What about $F_t(1-q)$, where

$$F_t(q) = q^t \sum_{k_t \geq \cdots \geq k_1 \geq 0}^{\infty} (q;q)_{k_t} \prod_{i=1}^{t-1} q^{k_i(k_i+1)} \begin{bmatrix} k_{i+1} \\ k_i \end{bmatrix}$$

is the generalized Kontsevich-Zagier series?

The coefficients appear to be positive.

What are they counting?

Thanks!