St Petesburg Oct 15, 2020

Mixed Hodge numbers and factorial ratios

Fernando Rodriguez Villegas
ICTP

Chebyshev

$$C_n = \frac{(30n)!\, n!}{(6n)!\,(10n)!\,(15n)!}$$

$n = 0, 1, \ldots$

is an integer.

Proof

$$v_p(C_n) = \sum_{k \geq 1} \eta\left(\frac{n}{p^k}\right)$$

$$\eta(x) := -\{30x\} - \{x\} + \{6x\} + \{10x\} + \{15x\}$$

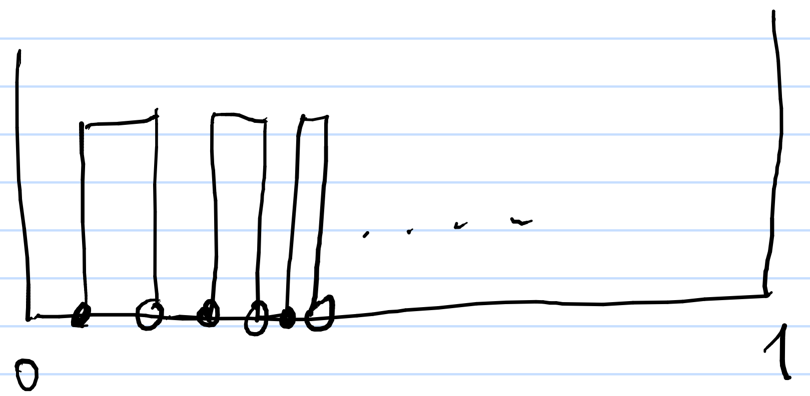
Intergrality follows from

$\eta(x) \geq 0$ all x

Landau proves it's actually equivalent.

$0 \leq \eta(x) \lesssim 1$

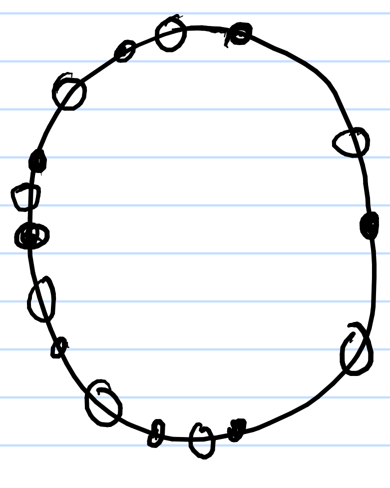
locally constant



Break points

• $\uparrow$ $1/30, 7/30, 11/30, 13/30, 17/30, 19/30, 23/30, 29/30$

∘ $\downarrow$ $0, 1/5, 1/3, 2/5, 1/2, 3/5, 2/3, 4/5$



interlace

$$\gamma = (-30, -1, 6, 10, 15)$$

$$\frac{(T^{30}-1)(T-1)}{(T^{6}-1)(T^{10}-1)(T^{15}-1)} = \frac{\Phi_{30}}{\Phi_1 \Phi_2 \Phi_3 \Phi_5}$$

α: $\quad e^{2\pi i \alpha_j}$

β: $\quad e^{-2\pi i \beta_j}$

$q_\infty, q_0 \in \mathbb{Z}[X]$
deg $= d$, coprime
roots interlace in the unit circle

Beukers–Heckman

Hypergeometric fctns in one variable are algebraic

Sound elementary proof

$$c(t) = \sum_{n \geq 0} c_n t^n \in \mathbb{Z}[[t]]$$

Algebraic function

$$A \in \mathbb{Z}[x, y]$$

$$A(t, c(t)) = 0$$

<u>Eisenstein</u>

$$1 + \sum_{n \geq 1} c_n t^n \in \mathbb{Q}[[t]]$$

$+$ algebraic $\Longrightarrow$ $c_n N^n \in \mathbb{Z}$

$\exists N \geq 1$

$$\gamma = (\gamma_1, \ldots, \gamma_e)$$

$$\gamma_1 \leq \gamma_2 \leq \ldots \leq \gamma_e$$

- $\sum_{i=1}^{e} \gamma_i = 0$
- $\gamma_i \neq 0$, $\gamma_i + \gamma_j \neq 0$
- γ_i no common divisor > 1

$$\eta(x) := -\sum_{\gamma_i < 0} \{-\gamma_i x\} + \sum_{\gamma_i > 0} \{\gamma_i x\}$$

<u>THM</u> (Landau)

$$c_n = \frac{\prod_{\gamma_i < 0} (-\gamma_i n)!}{\prod_{\gamma_i > 0} (\gamma_i n)!}$$

$\in \mathbb{Z}$ all n

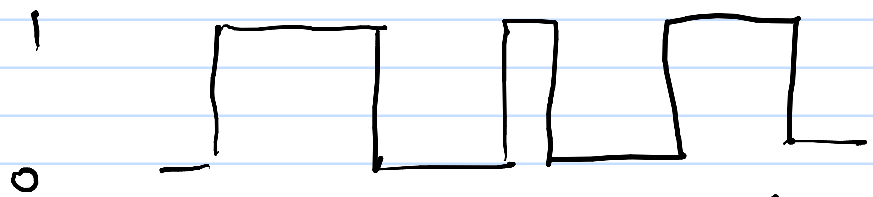
$\iff \quad \eta(x) \geq 0$

<u>Previous pblm</u>

$0 \leq \eta(x) \leq 1$

locally constant

$\vdots$



$\iff$ α's, β's interlace

γ gamma list

<u>Example</u>

$r \lesssim s$

$$r = \#\{\gamma_i < 0\}$$

$$s = \#\{\gamma_i > 0\}$$

$$\ell = r + s$$

$\boxed{r = 1}$

$$a = (a_1, \dots, a_s) \qquad a_i > 0$$

$$|a| = a_1 + \dots + a_s \qquad \text{primitive}$$

$$c_n = \frac{(|a|n)!}{(a_1 n)! \dots (a_s n)!}$$

$$A \in \mathbb{Z}^{(s-1)\times s} \qquad \operatorname{rk} A = s$$

$$A a = 0 \qquad \ker A = \mathbb{Z} a$$

$$A = (m_1, \dots, m_s) \quad \text{columns}$$

$$P(x) := \sum_{i=1}^{s} x^{m_i}$$

$$x = (x_1, \dots, x_{s-1})$$

$$\frac{1}{(2\pi i)^{s-1}} \int_{|x_j|=1} \frac{1}{1-\lambda P(x)} \frac{dx_1}{x_1} \cdots \frac{dx_{s-1}}{x_{s-1}}$$

$$= \sum_{n \geq 0} [P^n(x)]_0 \, \lambda^n$$

$[\cdot]_0 =$ constant term

$$= \sum_{n \geq 0} c_n \lambda^{|a| n}$$

$$\boxed{X_\lambda : \; 1 - \lambda P(x) = 0 \subseteq (\mathbb{C}^\times)^{s-1}}$$

algebraic variety

c is a period of X_λ

$$= \int_\Gamma \omega \qquad \text{Residue}$$

$\omega =$ differential form on X_λ

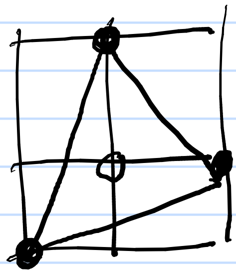
Example

$a = (1,1,1) \qquad |a| = 3$

$$c_n = \frac{(3n)!}{n!^3}$$

$$c(t) = \sum_{n \geq 0} \frac{(3n)!}{n!^3} \left(\frac{t}{27}\right)^n$$

period of X_λ $\qquad \Delta$:



$$A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}$$

$$P(x,y) = x + y + x^{-1}y^{-1}$$

X_λ : $\quad 1 - \lambda(x + y + x^{-1}y^{-1})$

$$xy - \lambda(x^2y + xy^2 + 1)$$

elliptic curves

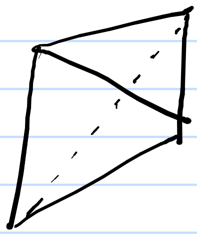
$\underline{r > 1}$

$\underline{r = 2}$ $\qquad \gamma = (\overbrace{-30, -1}^{r}, \overbrace{6, 10, 15}^{s})$

$$m_1, \dots, m_\ell \in \mathbb{Z}^{\ell-2}$$

$\mathbb{Z}^{\ell-2}$ $\qquad \ell-2$ vectors
lin indep. generically

$\ell-1$ vectors
affine lin indep
generically

Δ 

simplex

$\ell-2$ vectors

unique affine relation

want: $m_1, \dots, m_{\ell-2}$
s.t $\gamma = (\gamma_1, \dots, \gamma_{\ell-2})$

spans this

$$\Delta = \mathrm{Conv}(m_1, \dots, m_{\ell-2})$$

$$F_u := \sum_{i=1}^{\ell} u_i X^{m_i} \qquad u_i \in \mathbb{C}^\times$$

$$X = (X_1, \dots, X_{\ell-2})$$

scale $x_i \longmapsto a_i x_i \quad i=1,\dots,l-2$

$F_u \longmapsto a_0 F_u$

$l-1$ scalings

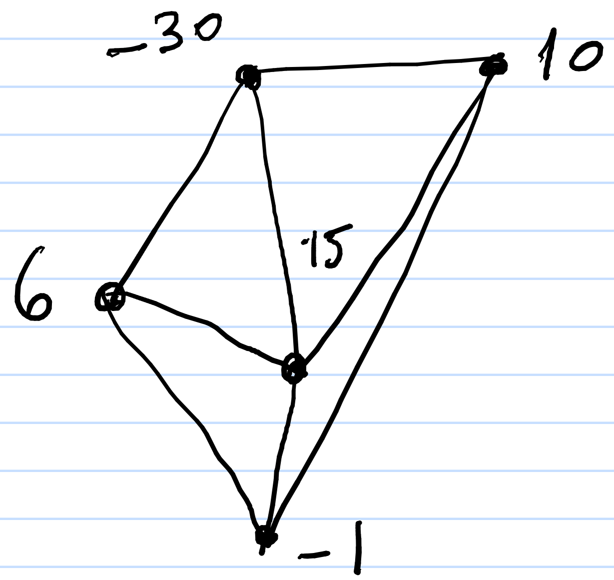
$\rightsquigarrow$ 1-dim'l family of Laurent polynomials

<u>Chebyshev</u>

$$\begin{pmatrix} 1 & 0 & 5 & 0 & 0 \\ 1 & 0 & 0 & 3 & 0 \\ 1 & 0 & 0 & 0 & 2 \end{pmatrix}$$

$-30 \quad -1 \quad 6 \quad 10 \quad 15$

Δ:



$$Z_t: \; xyz - \frac{M}{t} + x^5 + y^3 + z^2 = 0$$

$t = 0, 1, \infty$

$M = 30^{30}/6^6 \cdot 10^{10} \cdot 15^{15}$ singularities

Z_t : a family of algebraic surfaces

Equivalence between $c_u \in \mathbb{Z}$ for all u and geometric statement about Δ and about $Z_t : F_u = 0$

1) $k\Delta \quad k = 1, 2, \ldots, r-1$

no lattice interior points

2) $Z_t : F_u = \sum_{i=1}^{\ell} u_i x^{m_i} = 0 \subseteq (\mathbb{C}^\times)^{\ell-2}$

$x = (x_1, \ldots, x_{\ell-2})$

$\dim Z_t = \ell - 3$

$H_c^{\ell-3}(Z_t, \mathbb{Q})$

$\otimes \mathbb{C}$ Mixed Hodge decomposition

piece of weight $\ell - 3$

Hodge vector

$$h = (\underbrace{h^{\ell-3,0}, h^{\ell-4,1}}_{\underbrace{0 \cdots 0}_{r-1}}, \dots, \underbrace{h^{0,\ell-3}}_{\underbrace{0 \cdots 0}_{r-1}})$$

$\ell - 3 = 2 \qquad r = 2$

$$\underset{0}{h^{2,0}}, h^{1,1}, \underset{0}{h^{0,2}}$$

$h^{\ell-3,0}$ = geometric genus

$r = 2$ Cn integrality

$\Longleftrightarrow$ geometric genus $= 0$

<u>Examples</u>

1) $\gamma = (\underbrace{-11, -2}_{2}, \underbrace{1, 3, 4, 5}_{4})$

Z_t dim 3

can choose m_i's

F_u is a cubic $\subseteq (\mathbb{C}^\times)^4$

$\subseteq \mathbb{P}^4$

smooth cubic 3-folds

$t = 0, 1, \infty$

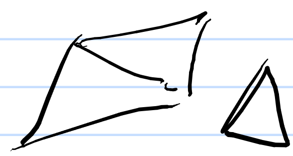
H^3 Hodge numbers

$(0, 5, 5, 0)$

$\uparrow$

2) $\gamma = (\underbrace{-63, -8, -2}_{3}, \underbrace{1, 4, 16, 21, 31}_{5})$

$\ell = 8$



Z_t 5-fold

cubic

$(\underbrace{0, 0}_{2}, 21, 21, \underbrace{0, 0}_{2})$