


Lecture 6 : 20 October 2020

I) Review

II) Hecke-type double-sums

and false theta functions

III) intro to quantum modular forms

IV) Radial limits of mock theta functions

Lecture 6 (detailed)

I) Review

First five lectures

II) Hecke-type double-sums $\left\{ \begin{array}{l} \text{false theta functions} \\ \text{gab, c} \end{array} \right.$

$$\text{gab, c } (x, q, z) := \left(\sum_{r,s \geq 0} + \sum_{r,s < 0} \right) (-1)^{r+s} x^r y^s q^{rs} a(z) \text{, but } c(z)$$

$$= g(q), \quad \sum_{n=0}^{\infty} d(n) q^n$$

• false theta funs

III) intro quantum modular forms

- basic definition

- examples

$$\circ \sigma(q)$$

• expressions from radial limits

- graphical visualization

Lecture 6 (detailed)

IV) Radial Limits and mock theta functions

- Zudilin's proof
- unilateral \nsubseteq bilateral q-series
- use of heuristic to complete
a unilateral series to a bilateral
series \nsubseteq produce more radial limit
examples

Notation $q \in \mathbb{C}, 0 < |q| < 1$

$$(x)_{\infty} = (x;q)_\infty = \prod_{i=0}^{\infty} (1 - q^i x)$$

$$j(z;q) = \sum_{n=-\infty}^{\infty} (-1)^n z^n q^{\binom{n}{2}}$$

$$\text{TP}_d = (z;q)_\infty (q/z;q)_\infty (q;q)_\infty$$

$$\overline{J}_{a,m} := j(q^a z^m) \quad \overline{J}_{a,m} := j(-q^a z^m)$$

$$\overline{J}_m := \overline{J}_{m,3m} = \prod_{i=0}^{\infty} (1 - q^{mi})$$

Partial Theta function $\sum_{n=0}^{\infty} (-1)^n z^n q^{\binom{n}{2}}$

False theta function $\sum_{n=-\infty}^{\infty} \text{sg}(n) (-1)^n z^n q^{\binom{n}{2}}$

$$\text{sg}(n) := \begin{cases} 1 & n \geq 0 \\ -1 & n < 0 \end{cases}$$

$$m(x, q, z) := \frac{1}{j(z;q)} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{\binom{n}{2}} z^n}{1 - q^{n+1} x z}$$

I) Review

i) Lecture 1

Ramanujan's Lost Notebook, q-series,
and mock theta fns

- Mock theta functions have many forms
- Changing between forms has been a way to understand the functions
- classical problems
 - formulas for Fourier coefficients
 - Proving identities between functions
 - Ramanujan's major dep'n & Radial Limits
 - Modular transformation properties

I) Review

Lecture 2

Appell-Lerch functions, Hecke-type double-sums
and a heuristic

- $m(qx, q, z) = 1 - x m(x, q, z)$

$$m(x, q, z) = 1 - q^{-1} x m(q^{-1} x, q, z)$$

$$\vdots$$
$$m(x, q, z) \sim \sum_{r=0}^{\infty} (-1)^r x^r q^{-\binom{r+1}{2}}$$

- new identities for $m(x, q, z)$ function
- expand Hecke-type double-sums of type I symmetry in terms of $m(x, q, z)$ functions
- Application - new proofs of the six identities for Ramanujan's four tenth order mock theta functions. (RLN, Choi)

I) Review

Lecture 2

$$f_{a,b,c}(x,y,z) := \left(\sum_{r,s \geq 0} - \sum_{r,s < 0} \right) \begin{cases} r > s \\ x^r y^s z^t \end{cases}$$

$$f_{1,2,1}(x,y,z) = j(yz) m(z^2 x / y^2, z^3, -)$$

$$+ j(xz) m(z^2 y / x^2, z^3, -)$$

$$\frac{-y\sqrt[3]{z} j(-xyyz)}{\sqrt[3]{z} j(-yz^2/xz^3) j(-xz^2/yz^3)}$$

I) Review

Lecture 3

Kronecker-type identities with applications to classical results on sums of squares and sums of triangular numbers

(Kronecker) For $x, y \in \mathbb{Q} \setminus \{0\}$ or $|y| < |x|, |y| < 1$

$$\left(\sum_{r,s \geq 0} - \sum_{r,s \leq 0} \right) q^{rs} x^r y^s = \frac{j^3}{j(xjq) j(yjq)}$$

$$x, y \rightarrow -1$$

Lagrange's Four-Square Theorem

$$x, y^2 \rightarrow -1$$

Fermat's Two-Square Theorem

I) Review

Lecture 3

using heuristic we found a higher-dimensional analog of Kronecker's 1d

$$\left(\sum_{r,s,t \geq 0} + \sum_{r,s,t < 0} \right) g^{\text{rsrt+st}} x^r y^s z^t$$

$$= \sum_j \neq m' + \text{theta}$$

$$x, y, z \rightarrow -1$$

Legendre-Gauss Local-to-global principle

$$r_3(n) > 0 \iff n \neq 4^a(8b+7)$$

$$g \rightarrow g^2, x = y = z \rightarrow g$$

Gauss's Eureka Theorem

$$num = 1 + 1 + 1, \Delta = \frac{1 \times (1+1)}{2}$$

I) Review

Lectures 4 $\frac{1}{2}$ 5

Two Problems

- Simultaneous representation of primes by
by binary quadratic forms (Kaplansky)
- A q-series from the Lost Notebook
and two conjectures of Andrews

$$\begin{aligned} S(q) &= 1 + \sum_{n=1}^{\infty} q^{n(n+1)/2} / ((1+q)(1+q^2)\cdots(1+q^n)) \\ &= \sum_{n=0}^{\infty} S(n)q^n \end{aligned}$$

$$- \limsup_{n \rightarrow \infty} |S(n)| = +\infty$$

$$- S(n) = 0 \text{ for infinitely many } n$$

I) Review

Lectures 4 & 5

underlying theory and questions

- Threefold identities
- Relating the Fourier coefficients at a q -series $\sum_{n=0}^{\infty} c(n)q^n$ to the solutions of a binary quadratic form $ax^2 + by^2 = z^2$
- When can one write a q -series as

$$f_{a,b,c}(x,y,z) := \left(\sum_{r,s \geq 0} - \sum_{r,s \leq 0} \right) (-1)^{r+s} \frac{a(r)}{z} + b r s + c \left(\sum_{r,s \geq 0} \right)$$

$$g_{a,b,c}(x,y,z) := \left(\sum_{r,s \geq 0} + \sum_{r,s \leq 0} \right) (-1)^{r+s} \frac{a(r)}{z} + b r s + c \left(\sum_{r,s \geq 0} \right)$$

Type I, Type II symmetry.

Lecture 6 (Today)

- I) more on Hecke-type double-sums
with type II symmetry and
false theta functions
- II) introduction to quantum modular forms
examples - $\sigma(g)$ ADH, C
- Radial limits of mock theta
functions
- III) Radial limits of mock theta functions
 - Zudilin's proof of Ramanujan's Claim
 $\frac{1}{2}$ role of bilateral q -hypergeometric series
 - use of heuristic to produce more bilateral
series and more radial limit results

I) Hecke-type double-sums w/ type II

symmetry $\{$ false theta functions

Previously

$$f_{a,b,c}(x, \gamma, z) := \left(\sum_{r,s \geq 0} - \sum_{r,s < 0} \right) (-1)^{r+s} x^r \gamma^s q^z a(\gamma) + b\gamma s + c(z)$$

$$g_{a,b,c}(x, \gamma, z) := \left(\sum_{r,s \geq 0} + \sum_{r,s < 0} \right) (-1)^{r+s} x^r \gamma^s q^z a(\gamma) + b\gamma s + c(z)$$

$$f_{1,2,1}(q, \bar{q}, q) = \bar{J}_1^2$$

$$f_{1,2,1}(q, -\bar{q}, q) = \bar{J}_{1,4} \phi(q)$$

$$\text{where } \phi(q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{(n+1)n}}{(-q;q)_\infty} = 2 \pi i (q_1 q_4)^{-1}$$

6th order mod η fn.

$$g_{a,b,c}(x, \gamma, q) \quad \sigma(q), \quad \sum_{n=0}^{\infty} \sigma(n) q^n$$

$$g_{1,2,2}(q, -\bar{q}, q) = -1 + \sum_{n=-\infty}^{\infty} \operatorname{sg}(n) (-2)^n q^{n(n+1)/2}$$

I) Hecke-type double-sum w/ type II symmetry
and false theta functions

$j(z; q)$ at $^{1/2}$

$m(x, y, z)$ at $^{1/2}$

false theta at $^{1/2}$

$f_{a,b,c}(x, y, q)$ examples weight one

garlic (x, y, q)

false theta have cut $^{1/2}$

Chan $\frac{1}{2}$ Kim

Andrews $\frac{1}{2}$ Warnaar

This also happens with $f_{a,b,c}(x, y, q)$
evaluations

I) Hecke-type double-sums w/ type II symmetry
and false theta functions

proof #1

$$f_{1,2n}(x, \gamma, q) = j(\gamma, q) m(q^2 x | \gamma^2, q^3, -)$$

$$+ j(x, q) m(q^2 \gamma | x^2, q^3, -)$$

$$- \overline{j_0} \overline{j_3} \overline{j(-x | \gamma, q)} j(q^2 x | \gamma, q^3)$$

$$\overline{j_0} \overline{j_3} \overline{j(-q\gamma^2 | x, q^3)} j(q^2 x | \gamma, q^3)$$

note $m(q, q^2, -1) = \frac{1}{2}$

$$m(-1, q^2, q) = 0$$

$$f_{1,2n}(-q, q, q^2) = J_{1,2} \cdot \frac{1}{2} + \overline{J}_{1,2} \cdot \frac{1}{2} + 0$$

$$= \overline{J}_{4,8} \text{ at } \frac{1}{2}$$

$$\text{used } j(z, q) = j(-q z^2, q^4) - z j(-q^3 z^2, q^4)$$

I) Hecke-type double-sums w/ type II symmetry
and false theta functions

proof #2 Functional equations

$$A) f_{a,b,c}(x, \gamma, q) = -q^{\frac{a+b+c}{2}} \frac{x^{\frac{a+b+c}{2}}}{xy} f_{a,b,c}\left(q^{\frac{2a+b}{2}}, q^{\frac{2c+b}{2}}, \gamma, x\right)$$

$$B) f_{a,b,c}(x, \gamma, q) \\ := (-1)^{\frac{a+b+c}{2}} x^{\frac{a+b+c}{2}} q^{\frac{a(\frac{l}{2}) + b(lk) + c(\frac{k}{2})}{2}} \\ f_{a,b,c}\left(q^{\frac{al-bk}{2}}, q^{\frac{bc+lk}{2}}, \gamma, x\right)$$

$$+ \sum_{r=0}^{l-1} (-1)^r x^{\frac{ra(\frac{r}{2})}{2}} j\left(q^{\frac{rb}{2}}, \gamma_0 q^{\frac{rc}{2}}\right)$$

$$+ \sum_{s=0}^{(l-1)} (-1)^s \gamma^s q^{\frac{s(s+1)}{2}} j\left(q^{\frac{sb}{2}}, \gamma_0 q^{\frac{sa}{2}}\right)$$

I) Hecke-type double-sums w/ type II symmetry
and false theta functions

proof #12 functional equations

In (B) set $(l, k) = (1, 1)$

$$f_{1,2,1}(q, -q, q^2) = -q^6 f_{1,2,1} \left(-\frac{q}{j}, \frac{q}{j}, q \right)$$

$$+ j(q; q^2) + j(-q; q^2)$$

use (A) $\sum J$ notation

$$f_{1,2,1}(q, -q, q^2) = -f_{1,2,1}(q, -q, q^2) + J_{1,2} + \overline{J}_{1,2}$$

or equivalently

$$f_{1,2,1}(q, -q, q^2) = \frac{1}{2} (J_{1,2} + \overline{J}_{1,2}) \\ = \overline{J}_{1,2}$$

where we have used

$$\overline{j}(z; q) = j(-q z^2; q) - z \overline{j}(-q^3 z^2; q)$$

I) Hecke-type double-sums w/ type II symmetry
and false theta functions

Chan & Kim

$$g_{2,2,2}(q^2, -q^3, q) = \frac{1}{1+q} \left(-1 + \sum_{n=-\infty}^{\infty} \text{sg}(n) (-1)^n q^{n(n+1)} \right)$$

$$g_{1,2,4}(q, -q^4, q) = \frac{1}{1-q} \left(-1 + \sum_{n=-\infty}^{\infty} \text{sg}(n) (-1)^n q^{n(n+1)/2} \right)$$

Andrews & Warnaar

$$g_{4,3,1}(q^2, q^2, q) = \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)/2}$$

$$g_{3,1,3}(q^2, -q^3, q) = \sum_{n=0}^{\infty} (-1)^n q^{3n(n+1)/2}$$

$$g_{1,0,1}(q^2, -q^4, q) = \sum_{n=0}^{\infty} (-1)^n q^{2n(n+1)}$$

I) Hecke-type double-sums w/ type II symmetry
and false theta functions

Proofs: Two functional equations

$$A) g_{a,b,c}(x, \gamma, q) = q^{\frac{a+b+c}{2}} \frac{g_{a,b,c}(q^{2a+b}, q^{2c+b})}{x\gamma} g_{a,b,c}(q^{2a+b}, x, \gamma, q)$$

$$\begin{aligned} B) & g_{a,b,c}(x, \gamma, q) \\ & := (-1)^{\frac{a+b+c}{2}} x^{\frac{a}{2}} \gamma^{\frac{b}{2}} q^{\frac{a(\frac{b}{2}) + b(\frac{c}{2}) + c(\frac{a}{2})}{2}} \\ & \quad \cdot g_{a,b,c}(q^{al-blk}, q^{bc-lk}, q^{cl-bk}) \\ & + \sum_{r=0}^{l-1} (-1)^r x^{\frac{ra(\frac{r}{2})}{2}} \sum_{s \in \mathbb{Z}} s g(s) (-1)^{\frac{rs}{2}} \gamma^{\frac{bs}{2}} q^{\frac{c(\frac{rs}{2})}{2}} \\ & + \sum_{s=0}^{(l-1)} (-1)^s \gamma^{\frac{sc(\frac{s}{2})}{2}} \sum_{r \in \mathbb{Z}} s g(r) (-1)^{\frac{rs}{2}} x^{\frac{ra(\frac{r}{2})}{2}} q^{\frac{bs}{2}} \\ & - 2 \sum_{r=0}^{l-1} \sum_{s=0}^{k-1} (-1)^{r+s} x^{\frac{ra(\frac{r}{2}) + b(s+\frac{k}{2}) + c(\frac{s}{2})}{2}} \end{aligned}$$

$$sg(n) = \begin{cases} 1 & n \geq 0 \\ -1 & n < 0 \end{cases}$$

I) Hecke-type double-sum w/ type II symmetry and false theta functions

Summation convention

$$\begin{matrix} \text{if } b < a \\ b \end{matrix} \sum_{r=a}^{a-1} c_r = - \sum_{r=b+1}^{} c_r$$

consequence

$$\sum_{r=0}^{-1} c_r = - \sum_{r=0}^{} c_r = 0$$

I) Hecke-type double-sums w/ type II symmetry
and false theta functions

Example

$$g_{1,3,1}(q, q_1^2, q_2) = \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)/2}$$

Take in eqn (B) with $(l, k) = (0, 1)$

$$\begin{aligned} & g_{1,3,1}(q, q_1^2, q_2) \\ &= -q^2 g_{1,3,1}(q_1^4, q_1^3, q_2) + \sum_{r \in \mathbb{Z}} \operatorname{sg}(r) (-1)^r q^{(r+1)} \\ &= -q^2 g_{1,3,1}(q_1^4, q_1^3, q_2) + 2 \sum_{r=0}^{\infty} (-1)^r q^{(r+1)} \\ &= -g_{1,3,1}(q, q_1^2, q_2) + 2 \sum_{r=0}^{\infty} (-1)^r q^{(r+1)} \end{aligned}$$

by Th. eqn (A)

Compare extreme ends

$$g_{1,3,1}(q, q_1^2, q_2) = \sum_{r=0}^{\infty} (-1)^r q^{(r+1)}$$

I) Hecke-type double-sums of type II symmetry and false theta functions

Summary

- what are building blocks for Hecke-type double-sums of type II symmetry?
- can we say something new about $G(q)$?
or $\sum_{n=0}^{\infty} A(n)q^n$?

II) intro to quantum modular forms

A modular form is a holomorphic function f on the complex upper-half plane $\mathbb{H} = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$ satisfying the transformation equation

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z) \quad (*)$$

for all $z \in \mathbb{H}$ and all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \subseteq \operatorname{SL}_2(\mathbb{Z})$

- - -

many variations >

- $k \in \frac{1}{2} \mathbb{Z}$
- $\Gamma = \operatorname{SL}_2(\mathbb{Z})$, $\Gamma = \Gamma_0(N) = \left\{ \gamma \in \operatorname{SL}_2(\mathbb{Z}) \mid \gamma \equiv \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \pmod{N} \right\}$
- f could be vector-valued
- meromorphic instead of holomorphic
- f may need correction term

Ex: Eisenstein Series

Theta functions

→ only need to check $(*)$ on generators of Γ

II) intro to quantum modular forms

A modular form is a holomorphic function f on the complex upper-half plane $\mathbb{H} = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$ satisfying the transformation equation

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z) \quad (*)$$

for all $z \in \mathbb{H}$ and all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \subseteq \operatorname{SL}_2(\mathbb{Z})$

- - -

rewrite condition (*)

$$f(z) - \varepsilon(\gamma)^{-1} (cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right) = 0$$

what about functions $f(z)$ where the right-hand side is not necessarily zero?

what kinds of conditions or requirements

should we place on the right hand side?

what about domain of f ?

II) intro to quantum modular forms

A modular form is a holomorphic function f on the complex upper-half plane $\mathbb{H} = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$ satisfying the transformation equation

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$$

for all $z \in \mathbb{H}$ and all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \subseteq \operatorname{SL}_2(\mathbb{Z})$

- - -

Let us focus on functions $g: \mathbb{Q} \rightarrow \mathbb{C}$

$$\text{and } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \subseteq \operatorname{SL}_2(\mathbb{Z})$$

what conditions do we put on

$$g(x) - \varepsilon(\gamma)^{-1} (cx+d)^{-k} g\left(\frac{ax+b}{cx+d}\right) = ???$$

II) intro to quantum modular forms

Def (Zagier) (2010)

A quantum modular form of weight k ($k \in \frac{1}{2}\mathbb{Z}$)

is a function $g: \mathbb{Q} \setminus S \rightarrow \mathbb{C}$ for some discrete subset S such that for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$

the functions

$$h_\gamma(x) := g(x) - \epsilon(\gamma)^{-1} (cx+d)^{-k} g\left(\frac{ax+b}{cx+d}\right)$$

satisfies a suitable property of continuity
or analyticity in \mathbb{R} .

Remark The definition was left vague

to allow for more examples.

II) intro to quantum modular forms

- Kontsevich's strange function

$$F(q) = \sum_{m=0}^{\infty} (1-q)(1-q^2)\dots(1-q^m)$$

- Andrews, Dyson, Hickerson $\psi(q)$ function
Cohen

- generating functions for strongly unimodal sequences (Bryson, Ono, Pittman, Rhoades)

- expressions related to radial limits
of mock theta function)
(Folsom, Ono, Rhoades)

(Kim, Lim, Lovejoy)

(Folsom, et al.)

II) intro to quantum modular forms

Andrews, Dyson, Hickerson, Cohen $\sigma(q)$ fn

we have two series

$$\begin{aligned}\sigma(q) &= 1 + \sum_{n=1}^{\infty} q^{n(n+1)/2} / (1+q)(1+q^2) \cdots (1+q^n) \\ &= 1 + q - q^2 + 2q^3 - 2q^4 + q^5 - q^6 + \cdots\end{aligned}$$

$$\begin{aligned}\sigma^*(q) &= 2 \sum_{n=1}^{\infty} (-1)^n q^{n^2} / (1-q)(1-q^3) \cdots (1-q^{2n-1}) \\ &= -2q - 2q^2 - 2q^3 + 2q^7 + 2q^8 + \cdots\end{aligned}$$

Recall

$$\zeta(q) = q_{1,3,3}(-q, q^2, q) - q_1 q_{1,3,3}(-q, q, q)$$

$$\zeta^*(q) = -q_1 q_{1,3,3}(q^3, q^8, q) - q_1 q_{1,3,3}(q, q^5, q^{10})$$

$$g_{a,b,c}(x, q) := \left(\sum_{r,s \geq 0} + \sum_{r,s < 0} \right) (-1)^{r+s} x^r q^s$$

II) intro to quantum modular forms

Andrews, Dyson, Hickerson, Cohen $\sigma(q)$ fn

we have two series (rewritten)

$$\sigma(q) = 1 + \sum_{n=0}^{\infty} (-1)^n q^{n+1} (1-q)(1-q^2) \dots (1-q^n) \quad (A)$$

$$\sigma^*(q) = -2 \sum_{n=0}^{\infty} q^{n+1} (1-q^2)(1-q^4) \dots (1-q^{2n}) \quad (B)$$

- Right-hand side of both series converges $|q| < 1$
- " " " "

make sense when q is a root of unity

Fact Define σ and σ^* at roots of

unity by (A) \nmid (B). Then $\sigma(q) = -\sigma^*(q^{-1})$

for every root of unity q .

II) intro to quantum modular forms

Andrews, Dyson, Hickerson, Cohen $\sigma(q)$ fn

we have two series (rewritten)

$$\sigma(q) = 1 + \sum_{n=0}^{\infty} (-1)^n q^{n+1} (1-q)(1-q^2) \dots (1-q^n) \quad (A)$$

$$\sigma^*(q) = -2 \sum_{n=0}^{\infty} q^{n+1} (1-q^2)(1-q^4) \dots (1-q^{2n}) \quad (B)$$

Fact $\sigma(q) = -\sigma^*(q^{-1})$ for all roots of unity q

Examples $q=1$

$$\sigma(1) = 1 + 1 = 2, \quad \sigma^*(1) = -2$$

$$q = -1$$

$$\sigma(-1) = 1 + (-1) + -(-1)^2 (2) = -2$$

$$\sigma^*(-1) = -2(-1) = 2$$

$$\sigma(\omega) = -\sigma^*(\omega^2) = 2\omega + b \quad \omega^2 + \omega + 1 = 0$$

$$\sigma(\pm i) = -\sigma^*(\mp i) = \mp 2i - 4 \quad i^2 = -1$$

II) intro to quantum modular forms

Andrews, Dyson, Hickerson, Cohen $\sigma(q)$ fn

we have two series (rewritten)

$$\sigma(q) = 1 + \sum_{n=0}^{\infty} (-1)^n q^{n+1} (1-q)(1-q^2) \dots (1-q^n) \quad (A)$$

$$\sigma^*(q) = -2 \sum_{n=0}^{\infty} q^{n+1} (1-q^2)(1-q^4) \dots (1-q^{2n}) \quad (B)$$

The quantum modular form

Define $f(x): \mathbb{Q} \rightarrow \mathbb{C}$ by

$$f(x) := q^{\frac{1}{24}} \underbrace{\sigma(q)}_{\sigma^*(q)} = -q^{\frac{1}{24}} \sigma^*(q), \quad x \in \mathbb{Q}, q = e^{2\pi i x}$$

- equality follows from Fact
- numerical observations

- $f(x)$ is very erratic

$$- h(x) := \frac{1}{2x+1} f\left(\frac{1}{2x+1}\right) - e^{\frac{2\pi i}{2x+1}} f(x)$$

appears smooth or C^∞ for $x \neq -\frac{1}{2}$

II) intro to quantum modular forms

Andrews, Dyson, Hickerson, Cohen $\sigma(q)$ fn

we have two series (rewritten)

$$\sigma(q) = 1 + \sum_{n=0}^{\infty} (-1)^n q^{n+1} (1-q)(1-q^2) \dots (1-q^n) \quad (A)$$

$$\sigma^*(q) = -2 \sum_{n=0}^{\infty} q^{n+1} (1-q^2)(1-q^4) \dots (1-q^{2n}) \quad (B)$$

Python Plot (Zagier fig 1)

$$f(x) := q^{\frac{1}{24}} \sigma(q) = -q^{\frac{1}{24}} \sigma^*(q^{-1})$$

we plot the real part of $f(x)$ for rational $x \in [1.7, 1.1]$ with the denominator of $x \leq 100$.

→ talk thru python code

→ we see $f(x)$ is very erratic

→ compare plot with Zagier fig 1 RealImag

→ find more fns plots. (no one does plots)

II) intro to quantum modular forms

Andrews, Dyson, Hickerson, Cohen $\sigma(q)$ fn

we have two series (rewritten)

$$\sigma(q) = 1 + \sum_{n=0}^{\infty} (-1)^n q^{n+1} (1-q)(1-q^2) \dots (1-q^n) \quad (A)$$

$$\sigma^*(q) = -2 \sum_{n=0}^{\infty} q^{n+1} (1-q^2)(1-q^4) \dots (1-q^{2n}) \quad (B)$$

Python Plot (Zagier Fig 2)

$$f(x) := q^{124} \sigma(q) = -q^{124} \sigma^*(q^{-1})$$

$$h(x) := \frac{1}{2x+1} f\left(\frac{1}{2x+1}\right) - e^{2\pi i / 24} f(x)$$

→ talk thru python code

→ we see $h(x)$ is smooth C^∞ except at $x = -\frac{1}{2}$

→ compare plot with figure 2 Zagier

→ find more fun plots

II) intro to quantum modular forms

Andrews, Dyson, Hickerson, Cohen $\sigma(q)$ fn

we have two series (rewritten)

$$\sigma(q) = 1 + \sum_{n=0}^{\infty} (-1)^n q^{n+1} (1-q)(1-q^2)\dots(1-q^n) \quad (A)$$

$$\sigma^*(q) = -2 \sum_{n=0}^{\infty} q^{n+1} (1-q^2)(1-q^4)\dots(1-q^{2n}) \quad (B)$$

Python Plot (zagier Fig 2)

$$f(x) := q^{12x} \sigma(q) = -q^{12x} \sigma^*(q^{-1})$$

$$h(x) := \frac{1}{2x+1} f\left(\frac{1}{2x+1}\right) - e^{2\pi i / 24} f(x)$$

Fact (zagier)

The fn $f: \mathbb{Q} \rightarrow \mathbb{C}$ satisfies

$$f(x+1) = e^{2\pi i / 24} f(x)$$

where $h: \mathbb{R} \rightarrow \mathbb{C}$ is C^∞ on \mathbb{R}

and real-analytic except at $x = -\frac{1}{2}$

II) intro to quantum modular forms

Andrews, Dyson, Hickerson, Cohen $\sigma(q)$ fn

we have two series (rewritten)

$$\sigma(q) = 1 + \sum_{n=0}^{\infty} (-1)^n q^{n+1} (1-q)(1-q^2) \dots (1-q^n) \quad (A)$$

$$\sigma^*(q) = -2 \sum_{n=0}^{\infty} q^{n+1} (1-q^2)(1-q^4) \dots (1-q^{2n}) \quad (B)$$

"leaking thru the rational numbers" (zagier)

$$f(z) := \begin{cases} q^{1/24} \sigma(q) & z \in H \cup \mathbb{Q} \\ -q^{1/24} \sigma^*(q^{-1}) & z \in H^- \cup \mathbb{Q} \end{cases}$$

$$H = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\} \quad H^- = \{z \in \mathbb{C} \mid \operatorname{Im} z < 0\}$$

f is analytic in both H , H^- and is

$\underset{\curvearrowleft}{\sim}$ on any curve passing vertically
through a rational pt.

III) Radial Limits of mock theta functions

Ramanujan's Last letter to Hardy

$$\begin{aligned}
 f(q) := \sum_{n=0}^{\infty} \frac{q^n}{\overline{(q;q)_n}^2} &= 1 + \frac{q}{(1+q)^2} + \frac{q^4}{(1+q)(1+q^2)^2} + \dots \\
 &= 1 + q - 2q^2 + 3q^3 - 3q^4 + 3q^5 - \dots
 \end{aligned}$$

- certain asymptotic properties as q approaches radially a root of unity
- similar to theta fun but not theta fun

III) Radial Limits of mock theta function >

Ramanujan's Last letter to Hardy

$$\begin{aligned}
 T(q) := \sum_{n=0}^{\infty} \frac{q^n}{\overline{(q;q)_n}} &= 1 + \frac{q}{(1+q)^2} + \frac{q^4}{(1+q)(1+q^2)^2} + \dots \\
 &= 1 + q - 2q^2 + 3q^3 - 3q^4 + 3q^5 - \dots
 \end{aligned}$$

- for every root of unity ζ there is a theta fn $\Theta_\zeta(q)$ such that the difference $T(q) - \Theta_\zeta(q)$ is bounded as $q \rightarrow \zeta$ radially
- there is no single theta fn that works for all ζ , i.e., for every theta fn $\Theta(q)$ there is a root of unity ζ for which $|T(q) - \Theta(q)|$ is unbounded as $q \rightarrow \zeta$ radially

III) Radial Limits of mock theta function >

Ramanujan's Last letter to Hardy

$$\begin{aligned} f(q) &:= \sum_{n=0}^{\infty} \frac{q^n}{\overline{(q;q)_n}^2} = 1 + \frac{q}{(1+q)^2} + \frac{q^4}{(1+q)(1+q^2)^2} + \dots \\ &= 1 + q - 2q^2 + 3q^3 - 3q^4 + 3q^5 - \dots \\ b(q) &:= \prod_{n=1}^{\infty} (1 - \frac{q^{2n-1}}{q})^2 (1 - q^n) = q^{1/2} / J_1 \end{aligned}$$

Ramanujan's Claim:

As q approaches a root of unity of order $2k$
the difference $|f(q) - (-1)^k b(q)|$ is
absolutely bounded

or

$$|f(q) - (-1)^k b(q)| = O(1)$$

III) Radial Limits of mock theta function >

Ramanujan's Last letter to Hardy

$$\begin{aligned}
 f(q) &:= \sum_{n=0}^{\infty} \frac{q^n}{\frac{1}{(-q;q)_n}} = 1 + \frac{q}{(1+q)^2} + \frac{q^4}{(1+q)(1+q^2)^2} + \dots \\
 &= 1 + q - 2q^2 + 3q^3 - 3q^4 + 3q^5 - \dots \\
 b(q) &:= \prod_{n=1}^{\infty} (1-q^{2n-1})^2 (1-q^n) = \frac{q^2}{J_{1,2} / J_1}
 \end{aligned}$$

Then (Folsom, Ono, Rhoades)

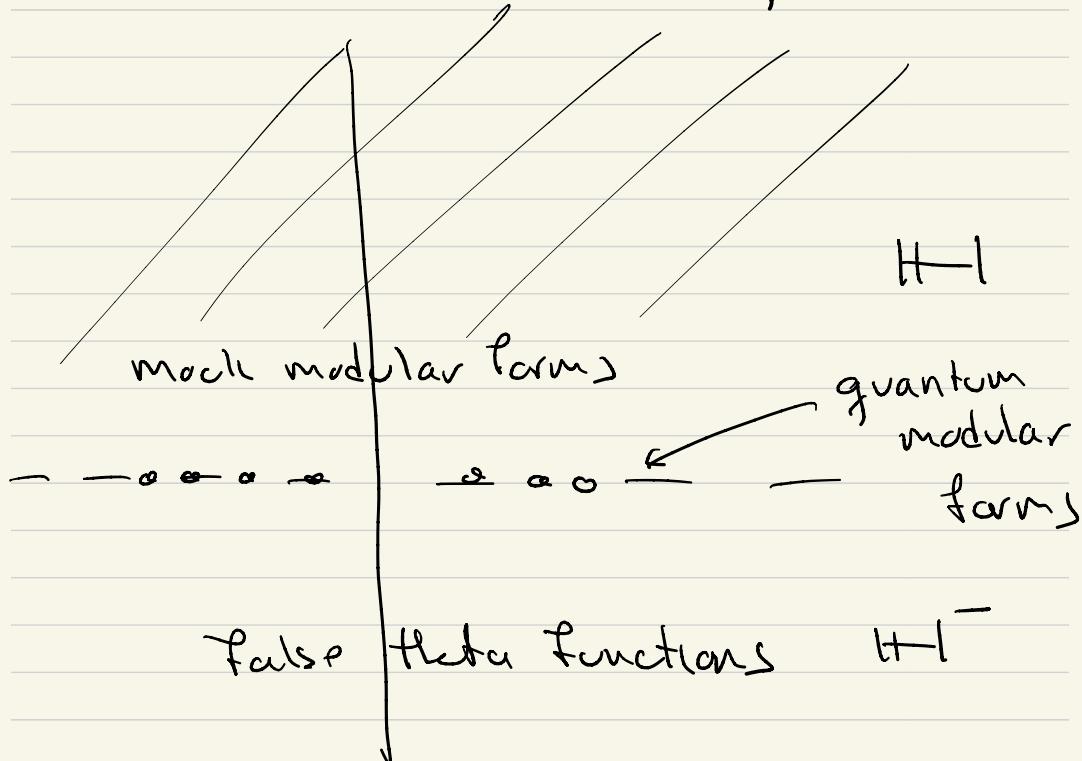
If ζ is a primitive even order $2k$ root of unity
 then as q approaches ζ radially within the unit disk we have

$$\begin{aligned}
 \lim_{q \rightarrow \zeta} f(q) - (-1)^k b(q) &= -4 \sum_{n=0}^{k-1} (1+\zeta)^2 (1+\zeta^2)^2 \dots (1+\zeta^n)^2 \zeta^{n+1}
 \end{aligned}$$

III) Radial limits of mock theta functions

what radial limits have to do with
quantum modular forms

$$\lim_{q \rightarrow 3^{-k}} f(q) - (-)^k b(q) = -4 \sum_{n=0}^{k-1} (1+3)^2 (1+3^2)^2 \dots (1+3^n)^2 3^{n+1}$$



FOR, Folsom Ono Rhoades

III) Radial Limits of mock theta function >

Ramanujan's Last letter to Hardy

$$\begin{aligned} f(q) := \sum_{n=0}^{\infty} \frac{q^n}{\overline{(q;q)_n}}^2 &= 1 + \frac{q}{(1+q)^2} + \frac{q^4}{(1+q)(1+q^2)^2} + \dots \\ &= 1 + q - 2q^2 + 3q^3 - 3q^4 + 3q^5 - \dots \end{aligned}$$

Folsom, Ono, Rhoades Theorem is a special case of a more general relation between generating functions for the rank and crank.

Proof involves weak harmonic Maass forms and mock modularity

Here we will give a q -series proof due to Wadim Zudilin

III) Radical Limits of mock theta functions

Zudilin's Proof of Ramanujan's claim

- discovered that all necessary ingredients for the proof are found in Ramanujan's Lost Notebook
- main ingredient is a bilateral q-series with mock modular behaviour.

Consequences

- one can use the heuristic to create bilateral q-series from unilateral q-series.
- with new bilateral q-series one can produce new radical limits for mock theta functions.

III) Radial Limits of Mock Theta Functions

From RLN

$$\sum_{n=-\infty}^{\infty} a^{-n-1} b^{-n} q^{n^2} / (-\gamma_{a;q})_{n+1} (-q/b;q)_n$$

$$= \frac{(-aq;q)_\infty}{b(q)_\infty (-q/b;q)_\infty} j(-b;q) m(a/b, q, -b)$$

want to write LHS as sum of two or laterals

$$\sum_{n=0}^{\infty} a^{-n-1} b^{-n} q^{n^2} / (-\gamma_{a;q})_{n+1} (-q/b;q)_n$$

$$+ \sum_{n=-\infty}^{-1} a^{-n-1} b^{-n} q^{n^2} / (-\gamma_{a;q})_{n+1} (-q/b;q)_n$$

$$(a)_{-n} = (-1)^n a^{-n} q^{n(n+1)/2} / (q/a;q)_n$$

III) Radial Limits of Mock Theta Functions

From RLN

$$\sum_{n=-\infty}^{\infty} a^{-n-1} b^{-n} q^{n^2} / (-a;q)_n (-b;q)_n$$

$$= \frac{(-aq;q)_{\infty}}{b(q)_{\infty} (-q/b;q)_{\infty}} j(-b;q) m(a/b, q, -b)$$

want to write LHS as sum of two unilateral

LHS

$$\sum_{n=0}^{\infty} a^{-n-1} b^{-n} q^{n^2} / (-a;q)_n (-b;q)_n$$

$$- \sum_{n=1}^{\infty} (-aq;q)_{n-1} (-b;q)_n q^n$$

$$= \sum_{n=0}^{\infty} a^{-n-1} b^{-n} q^n / (-a;q)_{n+1} (-q/b;q)_{n+1}$$

$$- (1+b) \sum_{n=0}^{\infty} (-aq;q)_n (-bq;q)_n q^{n+1}$$

III) Radial Limits of Mock Theta Functions

make substitution $a \rightarrow -\frac{1}{\omega}$ $b \rightarrow -\omega$

$$\begin{aligned} & \frac{\omega}{1-\omega} \sum_{n=0}^{\infty} q^n / (\omega q, q/\omega; q)_n \\ & - (1-\omega) \sum_{n=0}^{\infty} (\omega q, q/\omega; q)_n q^{n+1} \\ = & \frac{(q/\omega; q)_\infty j(\omega; q)}{\omega (q; q)_\infty (q/\omega; q)_\infty} m(\bar{\omega}^2, q, \omega) \end{aligned}$$

Recall

$$m(x, q, z) = x^{-1} m(x^{-1}, q, z^{-1})$$

$$\begin{aligned} & \sum_{n=0}^{\infty} q^n / (\omega q, q/\omega; q)_n \\ & + (1-\omega) (1-\bar{\omega}^{-1}) \sum_{n=0}^{\infty} (\omega q, q/\omega; q)_n q^{n+1} \\ = & (1-\omega) \frac{j(\omega, q)}{j_1} m(\bar{\omega}^2, q, \bar{\omega}^{-1}) \end{aligned}$$

Set $\omega = -1$

III) Radial Limits of mock theta functions

$$R(\omega; q) + (1-\omega)(1-\omega^{-1}) U(\omega; q)$$

$$= (1-\omega) j(\omega; q) \frac{\pi}{J_1} (\omega^2; q, \omega^{-1})$$

$$R(\omega; q) = \sum_{n=0}^{\infty} q^{n^2} / (q\omega; q, \omega; q)_n$$

$$U(\omega; q) = \sum_{n=0}^{\infty} (-q\omega; q, \omega; q)_n q^{n+1}$$

Let $\omega = -1$

$$R(-1; q) = f(q)$$

$$U(-1; q) = \sum_{n=0}^{\infty} (-q; q)_n q^{n+1}$$

$$f(q) + 4 \sum_{n=0}^{\infty} (-q; q)_n q^{n+1} = 2 \frac{J_{0,1}}{J_1} m(1, q, -1)$$

starting to look like FOR result

FOR RHS is terminating version of

III) Radial Limits of mock theta functions

$$f(q) + 4 \sum_{n=0}^{\infty} (-q;q)_n^2 q^{n+1} = 2 \frac{J_{0,1}}{J_1} m(1, q, -1)$$

↑

terminates when $q \rightarrow 3$,

3 primitive root of order $2k$

$$\sum_{n=0}^{k-1} (1+3)^2 (1+3^2)^2 \cdots (1+3^n)^2 3^{n+1}$$

because $1+3^k = 0$

$$f(q) = \sum_{n=0}^{\infty} q^n / ((-q;q)_n^2 = 1 + \frac{q}{(1+q)} + \frac{q}{(1+q)^2 (1+q^2)^2} + \cdots$$

$f(q)$ blows up when

$q \rightarrow 3$, 3 primitive root

order $2k$.

Q: What do we do with $m(1, q, -1)$? ??

A: Two more identities found in Lost Notebook!

III) Radical limits of moduli theta functions

$$(*) f(q) + 4 \sum_{n=0}^{\infty} (-q;q)_n^2 q^{n+1} = 2 \frac{J_{0,1}}{J_1} m(1, q_1 - 1)$$

$$A) \sum_{n=0}^{\infty} \frac{(-1)^n q^n (q;q)_n^2}{(-x;q)_n^2 (x;q)_n^2} = m(x, q_1 - 1) + \frac{J_{1,2}^2}{z_j(-x;q)}$$

$$B) \left(1 + \frac{1}{x}\right) \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)} (q;q)_n^2}{(-xq_1 - q/x; q)_n^2} = m(x, q_1 - 1) - \frac{J_{1,2}^2}{z_j(-x;q)}$$

$x=1$

$$A) \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-1)^n q^n (q;q)_n^2}{(-q;q)_n^2} = m(1, q_1 - 1) + \frac{J_{1,2}^2}{2 \bar{J}_{0,1}}$$

$$B) 2 \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)} (q;q)_n^2}{(-q;q)_n^2} = m(1, q_1 - 1) - \frac{J_{1,2}^2}{2 \bar{J}_{0,1}}$$

AHA!

III) Radial limits of mode theta functions

Rewrite (*) with λ

$$\ell(q) + 4 \sum_{n=0}^{\infty} (-q;q)_n^2 q^{n+1}$$

$$= 2 \frac{\bar{J}_{0,1}}{J_1} \left[\sum_{n=0}^{\infty} \frac{(-1)^n q^n (q;q)_n^2}{(-q;q)_n^2} - \frac{\bar{J}_{1,2}^2}{2\bar{J}_{0,1}} \right]$$

Rearrange

$$\ell(q) - \left(- \frac{\bar{J}_{1,2}^2}{\bar{J}_{0,1}} \right) = \frac{\bar{J}_{0,1}}{J_1} \sum_{n=0}^{\infty} \frac{(-1)^n q^n (q;q)_n^2}{(-q;q)_n^2}$$

↙
b!

$$-4 \sum_{n=0}^{\infty} (-q;q)_n^2 q^{n+1}$$

If $q \rightarrow \zeta_{2k}$ primitive $2k$ root of unity

k even both RHS & LHS have problems

k odd things look promising.

III) Radial limits of mock theta functions

$$P(q) - \left(-\frac{J_{12}^2}{J_1} \right) = \frac{\bar{J}_{01}}{J_1} \sum_{n=0}^{\infty} (-1)^n q^{\frac{n^2}{2}} \frac{(q;q^2)_n}{(-q^2;q^2)_n}$$

$$= -4 \sum_{n=0}^{\infty} (-q;q^2)_n^2 q^{n+1}$$

$q \rightarrow \zeta_{2k}$ primitive root of unity k odd

$$\frac{\bar{J}_{01}}{J_1} \rightarrow 0 \quad \text{one can show}$$

$$\sum_{n=0}^{\infty} (-1)^n q^{\frac{n^2}{2}} \frac{(q;q^2)_n}{(-q^2;q^2)_n} \xrightarrow{\text{lim. tr}}$$

$$\sum_{n=0}^{k-1} (-q;q^2)_n^2 q^{n+1} \xrightarrow{k-1} \sum_{n=0}^{k-1} \{(1+3)^2(1+3^2) - (1+3^n)^2\}^{n+1}$$

For $2k, k$ odd

$$\lim_{q \rightarrow \zeta_{2k}} \left(P(q) - (-1)^k \frac{J_{12}^2}{J_1} \right) = -4 \sum_{n=0}^{k-1} (-5;3)_n^2 q^{n+1}$$

k even use (B)

III) Radial limits of mock theta functions

- the bilateral series was key to Zudilin's proof
- given a unilateral series $\sum_{n=0}^{\infty} c(n) q^n$
can we complete it to a bilateral series?
- once we have the bilateral series, can we obtain more radial limit results?

Answers are both 'Yes'

We use the heuristic

$$m(x, q, z) \sim \sum_{r=0}^{\infty} (-1)^r x^r q^{-\binom{r+1}{2}}$$

IV) Radial limits and mock theta functions

Completing a unilateral series to a bilateral series

$$\sum_{n=0}^{\infty} \frac{(-1)^n (q;q)_n}{(-x;q)_{n+1} (-q/x;q)_n} = m(x, q, -1)$$

what is

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n (q;q)_n}{(-x;q)_{n+1} (-q/x;q)_n} = ?$$

or what is

$$\sum_{n=-\infty}^{-1} \frac{(-1)^n (q;q)_n}{(-x;q)_{n+1} (-q/x;q)_n}$$

$$= \left(1 + \frac{1}{x}\right) \sum_{n=0}^{\infty} \frac{(-q/x, -xq;q)_{n+1} q^n}{(q;q)_n}$$

use $(a)_{-n} = \dots$ and shift $n \rightarrow n+1$

III) Radial limits and mock theta functions

Completing a unilateral series to a bilateral series

hint: Zudlin's bilateral

$$\frac{1}{1-w} \sum_{n=0}^{\infty} q^{n^2} / \frac{(-wq;q)_n}{(-wq;q)_n} = 1 + w g_3(w; q)$$

$g_3(w; q)$ special expression involving m 's

$$(1-w) u(w; q) = -1 - w g_3(w; q) + \frac{w (w; q)_m (w^2; q)_j (-1)}{j}$$

$$u \sum_{n=0}^{\infty} \frac{(-1)^n (q; q^2)_n}{(-x)_{n+1} (-q/x)_n} = m(x, q, z)$$

assume

$$(1+x) \sum_{n=0}^{\infty} \frac{(-xq, -qx/xq)_n q^n}{(q; q^2)_{n+1}} = -m(x, q, z)$$

$$f(x) = m(x, q, z) \quad \text{or} \quad f(x) = -m(x, q, z)$$

(IV) Radial limits and mock theta functions

Completing a unilateral series to a bilateral series

$$f(x) := \left(1 + \frac{1}{x}\right) \sum_{n=0}^{\infty} \frac{(-xq_x, -q_x/x; q_x)^n q_x^n}{(q_x; q_x)^2 (n+1)}$$

$$\text{Assume } f(x) = -m(x, q_x, z)$$

$$\begin{aligned} f(q_x) &= -m(q_x, q_x, z) \\ &= -(1 - xm(x, q_x, z)) \\ &= -1 + xm(x, q_x, z) \end{aligned}$$

$$f(q_x) + 1 - xf(x) = 0$$

Is this true? No.

Numerically we have

$$f(q_x) + 1 - xf(x) = \frac{1}{x} \int \frac{(-x; q_x)^{\infty}}{(q_x; q_x)^2}$$

Assume \int is true.

Iterate it to see what heuristic tells us.

(IV) Radial limits and mock theta functions

Completing a unilateral series to a bilateral series

$$f(x) := \left(1 + \frac{1}{x}\right) \sum_{n=0}^{\infty} \frac{(-xq)_1 q^n / x; q^n}{(q; q)_n^2}$$

$$f(q_x) + (-x)f(x) = \frac{1}{x} \int \frac{(-x;q)_\infty}{J_{1,2}}$$

$$x \rightarrow q/x$$

$$f(x) = -1 + \frac{q}{x} \int \frac{(-x(q;q)_\infty)}{J_{1,2}} + \frac{x}{q} f\left(\frac{x}{q}\right)$$

$$\text{Iterate } \frac{1}{x} \text{ vs } - \int \left(\frac{q^n}{q} x; q\right)_\infty = (-1)^n q^{-\binom{n}{2}} x \int (x; q)_\infty$$

$$f(x) \sim - \sum_{r=0}^{\infty} (-1)^r x q^{r - \binom{r+1}{2}} + \int \frac{(-xq)_\infty}{J_{1,2}} \sum_{r=0}^{\infty} (-1)^r x q^{r + 2r - 2 \binom{r}{2}}$$

$$f(x) = -m(x, q, x)$$

$$+ \int \frac{(-xq)_\infty}{J_{1,2}} m(x^2, q^2, x) + \text{thd}q$$

(IV) Radical limits and mock theta functions

Completing a unilateral series to a bilateral series

$$f(x) := \left(1 + \frac{1}{x}\right) \sum_{n=0}^{\infty} \frac{(-xq, -q/x; q)_n q^n}{(q, q^2; q^2)_{n+1}}$$

turns out

$$\begin{aligned} f(x) &= -m(x, q, -1) + j(-xjq) m(x^2, q^2, -1) \\ &\quad - x \frac{j_4^3}{j_2^3} \frac{(xjq)}{j(q^2xjq)} \frac{j(q^2x^2jq^2)}{j(x^4jq^4)} \end{aligned}$$

↳ Bailey Technology

$$j_{1,2} f(x) = q F_{2,2,1} \left(q^3, -q^2; q^4 \right)$$

↳ then use $F_{2,2,1}(x, q, q)$ formula

can proceed as in Zudlin's Proof for
more radical limit results.

(IV) Radial limits and mock theta functions

Completing a unilateral series to a bilateral series

$$f(x) := \left(1 + \frac{1}{x}\right) \sum_{n=0}^{\infty} \frac{(-xq, -q/x; q)_n q^n}{(q, q^2; q^2)_{n+1}}$$

$$J_{1,2} f(x) = q^{\frac{3}{2}} f_{2,2,1}(q^{\frac{3}{2}}, -q^{\frac{1}{2}}; q, q)$$

$$\begin{aligned} f_{2,2,1}(x, y, q) &= j(y, q) m(qx(y^2, q^2, y^2)) \\ &\quad + j(x, q^2) m(-qy(x, q, x)) \end{aligned}$$

Singularities model of $m(x, q, z)$

need to use

$$\begin{aligned} m(x, q, z_1) &= m(x, q, z_0) \\ &\quad + z_0 \overline{j_1^3(z_1/z_0; q)} j(xz_0 z_1; q) \\ &\quad - \overline{j(z_0; q) j(z_1; q) j(xz_0 z_1; q)} j(xz_1; q) \end{aligned}$$

(IV) Radial limits and mock theta functions
Completing a unilateral series to a bilateral series

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n (q;q)_n^2}{(-x)_{n+1} (-q/x; q)_n} \\ = \frac{j(-aq)}{J_{1,2}} m(a^2, q^2, -1) - a \frac{\bar{J}_y^3}{J_2^3} \frac{j(ajq) j(qa^2; q^2)}{j(a^4; q^4)}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n (q;q)_n^2}{(-x)_{n+1} (-q/x; q)_n} + (1+x) \sum_{n=0}^{\infty} \frac{(-xq, -q/x)^n q}{(q;q^2)_{n+1}} \\ = \frac{j(-aq)}{J_{1,2}} m(a^2, q^2, -1) - a \frac{\bar{J}_y^3}{J_2^3} \frac{j(ajq) j(qa^2; q^4)}{j(a^4; q^4)}$$

instead another RLN is $\rightarrow z_0$ $\rightarrow z_1$

$$(1+x^{-1}) \sum_{n=0}^{\infty} q^n \frac{(-q)_n}{(qx, q/x; q^2)_{n+1}} = m(x, q^2, q)$$

$$m(x, q, z_0) = m(x, q, z_0) + z_0 \bar{J}_y^3 \frac{(z_1/z_0; q) j(xz_0 z_1; q)}{j(z_0; q) j(z_1; q) j(xz_0; q) j(xz_1; q)}$$

III) Radial limits and mock theta functions

Completing a unilateral series to a bilateral series

$$S_4(\omega; q) := \sum_{n=0}^{\infty} (-1)^n (q; q)_n / (-\omega q)_{n+1} (-q/\omega q)_n$$

Then if ζ is a primitive even order $2k$ root of unity then as q approaches ζ radially with the unit disk

$$\begin{aligned} \lim_{q \rightarrow \zeta} & \left(S_4(1;q) - \frac{\zeta^3}{\overline{\zeta}^2} - \frac{\overline{\zeta}^4}{2\overline{\zeta}^3} \right) \\ & = -2 \sum_{n=0}^{k-1} \zeta^{n+1} (-\zeta; \zeta)_n^2 / (\zeta; \zeta^2)_{n+1} \end{aligned}$$

Summary

I) Hecke-type double-sums type II symmetry

and false theta functions

$$g_{a,b,c}(q_1, q_2) := \left(\sum_{r,s \geq 0} r, s \atop \sum_{r,s \geq 0} r, s \right) (-1)^{r+s} q_1^{\binom{r}{2}} q_2^{\binom{s}{2}}$$
$$g_{4,3,1}(q_1, q_2, q) = \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)/2}$$

II) intro to quantum modular forms

- definitions

- python plots to demonstrate behaviour

- Andrews, Dyson, Hickerson, Cohen $G(q)$ fn.

III) Radial limits of mock theta functions

- Zudilin's proof

- use of heuristic to obtain

more results on radial limits.

References

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Next time

- mock modularity
- quantum modular forms
- modular forms