Lectures 4 & 5 & 6

Topics

I) Simultaneous representations of primes by binary quadratic forms

a) Kaplansky

II) A q-series from the Lost Notebook and two conjectures of Andrews

a) Andrews, Dyson, Hickerson

b) Cohen

III) An introduction to quantum modular forms

a) Zagier

b) Folsom, Ono, Rhoades

Notation

$q \in \mathbb{C},\ 0<|q|<1$

$$(x)_\infty = (x;q)_\infty = \prod_{i=0}^{\infty} (1-q^i x)$$

$$j(z;q) = \sum_{n=-\infty}^{\infty} (-1)^n z^n q^{\binom{n}{2}}$$

$$\text{JTP id} \quad = (z;q)_\infty (q/z;q)_\infty (q;q)_\infty$$

$$J_{a,m} := j(q^a;q^m) \qquad \bar{J}_{a,m} := j(-q^a;q^m)$$

$$J_m := J_{m,3m} = \prod_{i=1}^{\infty} (1-q^{im})$$

partial theta fn $\displaystyle\sum_{n=0}^{\infty} (-1)^n z^n q^{\binom{n}{2}}$

false theta fn $\displaystyle\sum_{n=-\infty}^{\infty} sg(n) (-1)^n z^n q^{\binom{n}{2}}$

$$sg(n) := \begin{cases} 1 & n \geq 0 \\ -1 & n < 0 \end{cases}$$

$$m(x,q,z) := \frac{1}{j(z;q)} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{\binom{n}{2}} z^n}{1-q^{n-1} x z}$$

Motivation:

I) Theorem (Kaplansky)

A prime $p$, $p \equiv 1 \pmod{16}$ is representable by both or none of the quadratic forms

$$x^2 + 32y^2 \quad \text{and} \quad x^2 + 64y^2$$

A prime $p$, $p \equiv 9 \pmod{16}$ is representable by exactly one of the quadratic forms

II) A $q$-series from the Lost Notebook

{ Andrews' Conjectures

$$G(q) := 1 + \sum_{n=1}^{\infty} q^{n(n+1)/2} \Big/ (1+q)(1+q^2)\cdots(1+q^n)$$

$$= \sum_{n=0}^{\infty} S(n) q^n$$

$$= 1 + q - q^2 + 2q^3 + \cdots + 4q^{45} + \cdots + 6q^{1609} + \cdots$$

Conjecture 1: $\limsup_{n \to \infty} |S(n)| = +\infty$

Conjecture 2: $S(n) = 0$ infinitely many $n$.

Underlying themes and questions

I) Threefield identities

a) Gauss ; Serre ; Andrews, Dyson, Hickerson; Cohen

II) Relating the Fourier coefficients of a q-series $\sum_{n=0}^{\infty} c(n) q^n$

to the solutions of a quadratic form

$$an+b = x^2 - dy^2$$

III) When can one write a q-series as

A) $$\left(\sum_{r,s\geq 0} - \sum_{r,s<0}\right)(-1)^{r+s} x^r y^s q^{a\binom{r}{2}+brs+c\binom{s}{2}}$$

B) $$\left(\sum_{r,s\geq 0} + \sum_{r,s<0}\right)(-1)^{r+s} x^r y^s q^{a\binom{r}{2}+brs+c\binom{s}{2}}$$

Underlying themes and questions

A) $f_{a,b,c}(x,y,q)$

$$:= \left(\sum_{r,s\geq 0} - \sum_{r,s<0}\right)(-1)^{r+s} x^r y^s q^{a\binom{r}{2}+brs+c\binom{s}{2}}$$

B) $g_{a,b,c}(x,y,q)$

$$:= \left(\sum_{r,s\geq 0} + \sum_{r,s<0}\right)(-1)^{r+s} x^r y^s q^{a\binom{r}{2}+brs+c\binom{s}{2}}$$

Examples

$$f_{1,2,1}(q,q,q) = \overline{J}_1^2$$

$$f_{1,2,1}(q,-q,q) = 2\overline{J}_{2,4}\, m(q,q^3,-1)$$

$$g_{1,2,2}(q,-q^3,q) = \sum_{n=-\infty}^{\infty} sg(n)(-1)^n q^{n(n+1)/2}$$

$$g_{1,5,1}(-q,-q,q) - q^2 g_{1,5,1}(-q^4,-q^4,q) = 6(q)$$

$$sg(n) := \begin{cases} 1 & n\geq 0 \\ -1 & n<0 \end{cases}$$

# Three field identities (Notation)

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## Legendre Symbol

Let $p$ be an odd prime number

$$\left(\frac{a}{p}\right) := \begin{cases} 1 & \text{if } a \text{ is a quadratic res mod } p \\ -1 & \text{if } a \text{ is a nonresidue mod } p \\ 0 & \text{if } a \equiv 0 \pmod{p} \end{cases}$$

Examples

a) $$\left(\frac{-1}{p}\right) = (-1)^{\frac{p-1}{2}} = \begin{cases} 1 & \text{if } p \equiv 1 \pmod 4 \\ -1 & \text{if } p \equiv 3 \pmod 4 \end{cases}$$

b) $$\left(\frac{2}{p}\right) = (-1)^{\frac{p^2-1}{8}} = \begin{cases} 1 & \text{if } p \equiv 1,7 \pmod 8 \\ -1 & \text{if } p \equiv 3,5 \pmod 8 \end{cases}$$

c) Fermat's two square theorem

$$p = x^2 + y^2 \iff p \equiv 1 \bmod 4 \left[\text{or } \left(\frac{-1}{p}\right) = 1\right]$$

# Three field identities (Notation)

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## Jacobi Symbol

- generalization of the Legendre Symbol
- For any integer $a$ and any positive integer $n$,

  $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$,

  The Jacobi Symbol is defined as the product of the Legendre Symbols

$$\left(\frac{a}{n}\right) := \left(\frac{a}{p_1}\right)^{\alpha_1} \left(\frac{a}{p_2}\right)^{\alpha_2} \cdots \left(\frac{a}{p_k}\right)^{\alpha_k}$$

# Threefield Identities

- Let $D$ & $E$ be distinct square-free integers not equal to 1 and let $F$ be the square-free product of $DE$
- There is an identity between representations of odd integers $n$ for which the Jacobi Symbols

$$\left(\frac{D}{n}\right) = \left(\frac{E}{n}\right) = \left(\frac{F}{n}\right) = 1$$

by quadratic forms associated with the fields

$$\mathbb{Q}(\sqrt{D}),\ \mathbb{Q}(\sqrt{E}),\ \mathbb{Q}(\sqrt{F})$$

- There are two types of threefield identities

# Threefield identities

Two types:

Type I)

The case $D, E, F$ are all positive

Example: $\sigma(q)$

$D=2, E=3, F=6$ (today)

Type $\underline{\text{II}}$)

The case two of the integers say $D, E$ are negative and one say $F$ is positive

Example:

Kaplansky's Theorem (last time)

$D=-1, E=-2$ $F=2$

# Threefield Identities

We have a q-series $\sum_{n=0}^{\infty} c(n) q^n$

The Fourier coefficients $c(n)$ are related to three different quadratic forms with each form having its own generating fn.

Type II) Two of the generating fns turn out to be theta fns. The identity is two theta fns expressed in terms of $J$'s and a Hecke-type double-sum $f_{a,b,c}(x,y,z)$

Example $D=-1$ $E=-2$ $F=2$

$$J_{1,4} J_{2,4} = J_{1,2} \overline{J}_{1,4}$$
$$= f_{1,3,1}(q^{3/4}, -q^{3/4}, -q^{1/2})$$

Application: Kaplansky's Theorem

Type I) Less understood

# Threefield identities

Type II ) Kaplansky's Theorem (last time)

$$\overline{J}_{1,4}\overline{J}_{2,4} = \overline{J}_{1,2}\overline{J}_{1,4}$$
$$= f_{1,3,1}\left(q^{3/4}, -q^{3/4}, -q^{1/2}\right)$$
$$=: \sum_{n=0}^{\infty} c(n) q^n$$

$D = -1, \quad E = -2, \quad F = 2$

we can relate the $c(n)$'s to the solution sets of three different quadratic forms

$$an + b = x^2 - dy^2$$

| | | |
|---|---|---|
| $8n+1 = x^2 + y^2$ | $d = -1$ | D |
| $8n+1 = x^2 + 2y^2$ | $d = -2$ | E |
| $8n+1 = x^2 - 2y^2$ | $d = 2$ | F |

# Threefield Identities

Kaplansky's Theorem & Understanding q-series

- State the threefield identity
- Prove the threefield identity
- Express $c(n)$ as a sum over the weighted solution sets to the three different quadratic forms
- Prove Kaplansky's Theorem by comparing the weighted solution sets
- Show that the weighted solution sets give the respective generating functions found in the threefield identity.

A B C

$$J_{1,4}\overline{J}_{2,4} = J_{1,2}\overline{J}_{1,4} = f_{1,3,1}(q^{3/4}, -q^{3/4}, -q^{1/2})$$

$$=: \sum_{n=0}^{\infty} c(n) q^n$$

A) Fix $x > 0$. The coeff $c(n)$ is the excess of the number of inequiv. solns of

$$8n+1 = x^2 + y^2$$

with

$$x \equiv \pm 1 \ (8),\ y \equiv 0 \ (8) \text{ or } x \equiv \pm 3 \ (8),\ y \equiv 4 \ (8)$$

over the number with

$$x \equiv \pm 3 \ (8),\ y \equiv 0 \ (8) \text{ or } x \equiv \pm 1 \ (8)\ y \equiv 4 \ (8)$$

B) Fix $x > 0$. Here $x$ is odd, $y$ even. The coeff $c(n)$ is the excess of the number of inequivalent solns of

$$8n+1 = x^2 + 2y^2$$

with

$$y \equiv 0 \ (4)$$

over the number with

$$y \equiv 2 \ (4)$$

A B C

$$J_{1,4}\overline{J}_{2,4} = J_{1,2}\overline{J}_{1,4} = f_{1,3,1}(q^{3/4}, -q^{3/4}, -q^{1/2})$$

$$=: \sum_{n=0}^{\infty} c(n) q^n$$

c) Fix $x > 0$, here $x$ is odd $y$ even and $-x/2 \le y < x/2$. The coeff $c(n)$ is the excess of the number of inequivalent solutions of

$$8n+1 = x^2 - 2y^2$$

with

$+$

$x \equiv 1 \ (4), y \equiv 0 \ (4)$ or $x \equiv 3 \ (4), y \equiv 2 \ (4)$

over the number with

$x \equiv 1 \ (4) \ y \equiv 2 \ (4)$ or $x \equiv 3 \ (4), y \equiv 0 \ (4)$

$-1$

Remark: inequivalent solutions.

# Threefield Identities

## Kaplansky's Theorem & Understanding q-series

Given a generating function

for example $J_{1,2} \cdot \overline{J}_{1,4} =: \sum_{n=0}^{\infty} c(n) q^n$

1) how do we find $a, b, d$ in

$$an+b = x^2 - dy^2 \ ? \quad (*)$$

2) once we have $a, b, d$, how do we express $c(n)$ as a sum over the weights of the solutions to $(*)$?

3) how do we show the weighted solution sets give the generating function?

# Threefield Identities

Kaplansky's Theorem & Understanding q-series

Finding a, b, d given a generating function

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For a theta function $f(q)$ there is an associated fractional exponent $\lambda$ such that $q^{\lambda} f(q)$ is in some ways simpler than $f(q)$. Modularity properties are easier to state. The $\lambda$ for both $J_{k,m} := j(q^k; q^m)$ and $\overline{J}_{k,m} := j(-q^k; q^m)$

is $\lambda = (m-2k)^2/8m$ "=" $a/b$

Remark — one does not necessarily assume a, b are in lowest terms

# Threefield identities

Kaplansky's Theorem & Understanding q-series

—o—

In $\overline{J}_{k_1,m} \cdot \overline{J}_{k_2,m}$

(or with $\overline{J}$'s)

$d = -$ (square-free part of $m_1 \cdot m_2$)

Example

$\overline{J}_{1,4}\overline{J}_{2,4}$ $\qquad 8n+1 = x^2+y^2$

$$\lambda = \frac{(4-2\cdot 1)^2}{8\cdot 4} + \frac{(4-2\cdot 2)^2}{8\cdot 4} = \frac{1}{8} \qquad d=-1$$

$J_{1,2}\overline{J}_{1,4}$ $\qquad 8n+1 = x^2+2y^2$

$$\lambda = \frac{(2-2\cdot 1)^2}{8\cdot 2} + \frac{(4-2\cdot 1)^2}{8\cdot 4} = \frac{1}{8}, \quad d=-2$$

# Threefield identities

## Kaplansky's Theorem & Understanding q-series

If we have the generating function and have a good guess at a, b, d, how do we find the weighted solution set to $an+b = x^2 - dy^2$ ?

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$$\sum_{n=0}^{\infty} c(n) q^n := \overline{J}_{1,2}\,\overline{J}_{1,4} = 1 - q - 2q^2 + q^3 + 2q^5 + q^6 + \cdots$$

Guess $\quad 8n+1 = x^2 + 2y^2$

a) Find n, $8n+1 = p$ a prime, $c(n) \neq 0$

These are p with $\left(\frac{-2}{p}\right) = 1$

$n=2 \quad c(2) = -2 \quad 8n+1 = 17$

Solns $(3,-2)$, $(3,2)$

give each weight "$-1$"

Python Code

# Threefield Identities

## Kaplansky's Theorem & Understanding q-series

b) For primes $p$, $8n+1=p$, $c(n)=0$

Look for $n$ with $8n+1=p^2$, $c(n)\neq 0$

primes $(-2/p)=-1$

$n=3$ $c(3)=1$ $8n+1=25$

solns $(5,0)$

give each weight "+1"

Pattern $c(n)$ is ...

$x>0$ odd $y$ even

$y\equiv 0 \pmod 4$ weight $+1$

$y\equiv 2 \pmod 4$ weight $-1$

# Threefield Identities

## Kaplansky's Theorem & Understanding q-series

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how do we show our guess at the weight system gives our generating function?

$$\sum_{n=0}^{\infty} a(n) q^n = \overline{J}_{1,2} \overline{J}_{1,4}$$

- group according to weights & not n's
- where do '+1''s come from

where do '-1''s come from

$$\sum_{x,y} w(x,y)\, q^{[x^2+2y^2-1]/8}$$

last time.

# Threefield identities

Kaplansky's Theorem & Understanding q-series

solns giving weight +1

$x = 2r+1 \qquad y = 4s \qquad x > 0$

solns giving weight −1

$x = 2r+1 \qquad y = 4s+2$

$$n = (x^2 + 2y^2 - 1)/8$$

+1 wts come from

$$\frac{1}{2} \sum_{r \in \mathbb{Z}} \sum_{s \in \mathbb{Z}} q^{[(2r+1)^2 + 2(4s)^2 - 1]/8}$$

−1 wts come from

$$-\frac{1}{2} \sum_{r \in \mathbb{Z}} \sum_{s \in \mathbb{Z}} q^{[(2r+1)^2 + 2(4s+2)^2 - 1]/8}$$

# Threefield Identities

Kaplansky's Theorem & Understanding q-series

$$\sum_{n=0}^{\infty} d(n) q^n$$

$$= \frac{1}{2} \sum_{r\in\mathbb{Z}} \sum_{s\in\mathbb{Z}} q^{[(2r+1)^2+2(4s)^2-1]/8}$$

$$- \frac{1}{2} \sum_{r\in\mathbb{Z}} \sum_{s\in\mathbb{Z}} q^{[(2r+1)^2+2(4s+2)^2-1]/8}$$

$$= \frac{1}{2} \sum_{r\in\mathbb{Z}} \sum_{s\in\mathbb{Z}} (-1)^s q^{[(2r+1)^2+2(2s)^2-1]/8}$$

$$= \frac{1}{2} \sum_{r} q^{r^2/2+r/2} \cdot \sum_{s} (-1)^s q^{s^2}$$

$$= \frac{1}{2} J_{0,1} J_{1,2}$$

$$= \overline{J}_{1,4} J_{1,2}$$

Today's Lecture.

A $q$-series from Ramanujan's Lost Notebook and two Conjectures of Andrews.

$$\sigma(q) := 1 + \sum_{n=1}^{\infty} q^{n(n+1)/2} \Big/ (1+q)(1+q^2)\cdots(1+q^n)$$

$$= \sum_{n=0}^{\infty} S(n) q^n$$

$$= 1 + q - q^2 + 2q^3 + \cdots + 4q^{45} + \cdots$$

Conjecture #1 $\quad \limsup_{n \to \infty} |S(n)| = +\infty$

Conjecture #2 $\quad S(n) = 0$ for infinitely many $n$

Theorem (Andrews, Dyson, Hickerson)

$S(n)$ is almost always zero, that is the set of $n$ for which $S(n) \neq 0$ has density zero. $S(n)$ takes on every integer value infinitely often.

Theorem (Andrews, Dyson, Hickerson)

$S(n)$ is almost always zero, that is the set of $n$ for which $S(n) \neq 0$ has density zero.

$S(n)$ takes on every integer value infinitely often

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Proof (Brief Sketch)

- ADH wrote $\sigma(q) = \sum_{n=0}^{\infty} S(n) q^n$ as a weighted sum over inequivalent solutions to

$$24n+1 = x^2 - 6y^2 \qquad (*)\ u^2 - 6v^2 = m$$

$T(m) :=$ no of inequiv. solns of $(*)$ with $u + 3v \equiv \pm 1 \ (12)$ over those with $u + 3v \equiv \pm 5 \ (12)$

- ADH showed $S(n) = T(24n+1)$
- Found an exact formula for $T(m)$
- Used formula for $T(m)$ to prove the theorem / conjectures.

# A $q$-series of RLN & Andrews two Conjectures

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Unlike our example where we related the Fourier coefficients of $J_{1,2}\overline{J}_{1,4} =: \sum_{n=0}^{\infty} c(n) q^n$ to all of the solutions of $8n+1 = x^2+2y^2$, we cannot write $T(m)$ as a sum of weights of all solutions to $u^2-6v^2=m$ because there are infinitely many!

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If $(u,v)$ is a soln to $u^2-6v^2=m$ $(*)$ and if $x^2-6y^2=1$ then

$$u'+\sqrt{6}v = (x+\sqrt{6}y)(u+\sqrt{6}v)$$

is also a solution to $(*)$

# A q-series of RLN & two Conjectures of Andrews

Fact:

Let $x, y$ be a solution to $x^2 - Dy^2 = 1$ where $x, y$ are minimal positive.

Example $D=6$ $x=5, y=2$ $25 - 6\cdot 4 = 1$

$D = 2$ $x = 3$ $y = 2$ $9 - 2\cdot 4 = 1$

Then each equivalent class of solutions to $u^2 - 6v^2 = m$ contains a unique $(u, v)$ with

$$u \geq 0 \quad \& \quad \frac{-y}{x+1} \cdot u < v \leq \frac{y}{x+1} u$$

Example $D = 6$ $u \geq 0$, $-\frac{u}{3} < v \leq u/3$

$D = 2$ $u \geq 0$ $-u/2 < v \leq u/2$

A q-series from RLN & Andrews two conjectures

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Given the q-series for $\sigma(q) = \sum_{n=0}^{\infty} S(n) q^n$ how do we know to relate the Fourier coefficients to the solutions of $24n+1 = x^2 - 6y^2$?

Q: How do we find a, b, d in $an+b = x^2 - dy^2$?

$$\sigma(q) = \sum_{n=0}^{\infty} S(n) q^n$$

$S(n) = 0$ for two reasons

1) the weights sum to zero

2) There are no solutions to $an+b = x^2 - dy^2$

# A q-series from RLN & two conjectures of Andrews

- If $p$ is a prime such that $d$ is a quadratic nonresidue $\left(\frac{d}{p}\right) = -1$ and if $p \mid x^2 - dy^2$ then $p^2 \mid x^2 - dy^2$

So if $p \mid an+b$ but $p^2 \nmid an+b$ then $S(n) = 0$ (no solutions to $an+b = x^2 - dy^2$)

So we test a bunch of primes

Write Fourier Coefficients in groups of $p$

| | | |
|---|---|---|
| $S(0)$ | $S(1)$ | $S(p-1)$ |
| $S(p)$ | $S(p+1)$ | $S(2p-1)$ |
| $S(2p)$ | $S(2p+1)$ | $S(3p-1)$ |
| $\vdots$ | | |

A $q$-series from RLN & two conjectures of Andrews

Look for a column with almost all zeros. with nonzero terms occuring a multiple of $p$ rows apart

If you find such a column for enough primes you should be able to guess $a, b$

A $q$-series from RCN & two conjectures of Andrews

| prime | col | index first $n$, $S(n) \neq 0$ |
|---|---|---|
| 5 | 1 | 1 |
| 7 | 2 | 2 |
| 11 | 5 | 5 |
| 13 | 7 | 7 |
| 17 | 12 | 12 |
| 19 | 15 | 15 |
| 29 | 6 | 35 |
| 31 | 9 | 40 |

let a range 1 to 100

let b range 1 to 100

look for (a,b) tuples where

$5^2 \mid a \cdot 1 + b$

$7^2 \mid a \cdot 2 + b$

$11^2 \mid a \cdot 5 + b$

$\vdots$

Find $a = 24$, $b = 1$

A $q$-series from RLN & two conjectures of Andrews

Once we have candidates for $a, b$ how do we find $d$?

Look for primes $p$ where $p = 24n+1$, $S(n) \neq 0$

The first 143 such primes are

$[73, 97, 193, \ldots, 9817]$

Because $S(n) \neq 0$ we must have $\left(\frac{d}{p}\right) = 1$ for each $p$.

The only possible $d$'s in the range $[-100, 100] \setminus \{0, 1\}$ are $2, 3, 6$.

A q-series from RLN & Andrews two Conjectures

—o—

Given the q-series $\sigma(q) = \sum_{n=0}^{\infty} S(n) q^n$

Once we have good cause to suspect the Fourier Coefficients $S(n)$ are related to the inequivalent solutions of $24n+1 = x^2 - 6y^2$,

how do we assign weights to the solutions?

$u^2 - 6v^2 = 24n + 1$ $\quad n \geq 0$ $\quad D = 6, \ x = 5, y = 2$

each equiv class of solns contains a unique

$(u, v)$ $\quad u > 0$ , $\quad -u/3 < v \leq u/3$

a congruence argument tells us

$u \equiv \pm 1 \ (6) \qquad v \equiv 0 \ (2)$

Pattern $\qquad 24n+1 = x^2 - 6y^2$

positive weights $\qquad +1$

$x \equiv 1 \ (12) \quad y \equiv 0 \ (4)$

$x \equiv 7 \ (12) \quad y \equiv 2 \ (4)$

$x \equiv 11 \ (12) \quad y \equiv 0 \ (4)$

$x \equiv 5 \ (12) \quad y \equiv 2 \ (4)$

negative wts $\qquad -1$

$x \equiv 7 \ (12) \quad y \equiv 0 \ (4)$

$x \equiv 1 \ (12) \quad y \equiv 2 \ (4)$

$x \equiv 5 \ (12) \quad y \equiv 0 \ (4)$

$x \equiv 11 \ (12) \quad y \equiv 2 \ (4)$

A $q$-series from RLN & two conjectures of Andrews

$u = 6i+1$ or $6i+5$

$v = 2j$

rewrite inequalities

$-u < 3v \le u$

$6i+1$: $-6i-1 < 6j \le 6i+1$

$-i - 1/6 < j \le i + 1/6$

so $|j| \le i$ $\quad i \ge 0$

$6i+5$: $-6i-5 < 6j \le 6i+5$

$-i - 5/6 < j \le i + 5/6$

$|j| \le i$ $\quad i \ge 0$

A $q$-series from RLN & two conjectures of Andrews

what is the pattern to the weights?

weight $+1$

$x \equiv 1 \pmod{12}$ $\quad y \equiv 0 \pmod 4$

$x \equiv 5$ " $\quad y \equiv 2$ "

$x \equiv 7$ " $\quad y \equiv 2$ "

$x \equiv 11$ " $\quad y \equiv 0$ "

weight $-1$

$x \equiv 1 \pmod{12}$ $\quad y \equiv 2 \pmod 4$

$x \equiv 5$ " $\quad y \equiv 0$ "

$x \equiv 7$ " $\quad y \equiv 0$ "

$x \equiv 11$ " $\quad y \equiv 2$ "

A q-series from RLN & Andrews two Conjectures

weight system → Hecke-type double sum

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$\sum$ means $\sum_{i\geq 0,\ |j|\leq i}$

$$\sum q^{[(12i+1)^2-6(4j)^2-1]/24} - \sum q^{[(12i+1)^2-6(4j+2)^2-1]/24}$$

$$-\sum q^{[(12i+5)^2-6(4j)^2-1]/24} + \sum q^{[(12i+5)^2-6(4j+2)^2-1]/24}$$

$$-\sum q^{[(12i+7)^2-6(4j)^2-1]/24} + \sum q^{[(12i+7)^2-6(4j+2)^2-1]/24}$$

$$+\sum q^{[(12i+11)^2-6(4j)^2-1]/24} - \sum q^{[(12i+11)^2-6(4j+2)^2-1]/24}$$

$$= \sum (-1)^i q^{[(6i+1)^2-6(4j)^2-1]/24} - \sum (-1)^i q^{[(6i+1)^2-6(4j+2)^2-1]/24}$$

$$- \sum (-1)^i q^{[(6i+5)^2-6(4j)^2-1]/24} - \sum (-1)^i q^{[(6i+5)^2-6(4j+2)^2-1]/24}$$

$$= \sum (-1)^{i+j} q^{[(6i+1)^2-6(2j)^2-1]/24} - \sum (-1)^{i+j} q^{[(6i+5)^2-6(2j)^2-1]/24}$$

$= \cdots$

A $q$-series from RLN & Andrews two Conjectures

weight system -> Hecke-type double-sum

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From Previous Page $\sum$ means $\sum_{i\geq 0,\, |j|\leq i}$

$$\sum (-1)^{i+j} q^{[(6i+1)^2-6(2j)^2-1]/24} - \sum (-1)^{i+j} q^{[(6i+5)^2-6(2j)^2-1]/24}$$

$$= \sum_{\substack{i\geq 0 \\ |j|\leq i}} (-1)^{i+j} q^{i(3i+1)/2-j^2} (1-q^{2i+1})$$

$$= \sum_{n=0}^{\infty} T(24n+1) q^n$$

Bailey Technology

$$G(q) = \sum_{\substack{i\geq 0 \\ |j|\leq i}} (-1)^{i+j} q^{i(3i+1)/2-j^2} (1-q^{2i+1})$$

so $\displaystyle\sum_{n=0}^{\infty} S(n) q^n = \sum_{n=0}^{\infty} T(24n+1) q^n$

$$\Rightarrow S(n) = T(24n+1)$$

A $q$-series from R&N & two conjectures of Andrews

$$G(q) = \sum_{\substack{i=0 \\ |j| \le i}}^{\infty} (-1)^{i+j} q^{i(3i+1)/2 - j^2} (1 - q^{2i+1})$$

want it to look like

$$\left( \sum_{r,s \ge 0} \pm \sum_{r,s < 0} \right) (-1)^{r+s} x^r y^s q^{a\binom{r}{2} + brs + c\binom{s}{2}}$$

let $r = i - j$ $\quad s = i + j$

then $i = \dfrac{r+s}{2}$ $\quad j = \dfrac{s-r}{2}$

need two cases

$$(r,s) \longrightarrow (2r, 2s) \quad \text{both even}$$

$$\hookrightarrow (2r+1, 2s+1) \quad \text{both odd}$$

A $q$-series from R&N & two conjectures of Andrews

case $(2r, 2s)$

$$\sum_{r,s\geq 0} q^{\binom{r}{2}+5rs+\binom{s}{2}+r+s}\left(1-q^{2r+2s+1}\right)$$

case $(2r+1, 2s+1)$

$$-\sum_{r,s\geq 0} q^{\binom{r}{2}+5rs+\binom{s}{2}+4r+4s+2}\left(1-q^{2r+2s+3}\right)$$

we have four sums above, in two of them make $r, s \to -1-r, -1-s$

$$G(q) = \Big(\sum_{r,s\geq 0} + \sum_{r,s<0}\Big) q^{\binom{r}{2}+5rs+\binom{s}{2}+r+s}$$

$$-q^2\Big(\sum_{r,s\geq 0} + \sum_{r,s<0}\Big) q^{\binom{r}{2}+5rs+\binom{s}{2}+4r+4s}$$

$$= g_{1,5,1}(-q, -q, q)$$

$$-q^2 g_{1,5,1}(-q^4, -q^4, q)$$

A $q$-series from RLN & two conjectures of Andrews

other generating fns for $\sigma(q)$

$$\sigma(q) = g_{3,6,6}(-q^2, q^4, q)$$
$$-q^4 g_{3,6,6}(-q^8, q^{10}, q)$$
$$+2q^3 g_{3,6,6}(-q^6, q^{10}, q)$$

Theorem (Andrews, Dyson, Hickerson)

$S(n)$ is almost always zero, that is the set of $n$ for which $S(n) \neq 0$ has density zero.

$S(n)$ takes on every integer value infinitely often

—∘—

Proof (Brief sketch)

- ADH wrote $\sigma(q) = \sum_{n=0}^{\infty} S(n) q^n$ as a weighted sum over inequivalent solutions to $24n+1 = x^2 - 6y^2$ $\qquad (*)\ u^2 - 6v^2 = m$

$T(m) :=$ no of inequiv. solns of $(*)$ with $u + 3v \equiv \pm 1\ (12)$ over those with $u + 3v \equiv \pm 5\ (12)$

- ADH showed $S(n) = T(24n+1)$
- Found an exact formula for $T(m)$
- Used formula for $T(m)$ to prove the theorem / conjectures.

# Proof of ADH Theorem

we know $S(n) = T(24n+1)$

$$\sigma(q) := 1 + \sum_{n=1}^{\infty} q^{n(n+1)/2} / (1+q)(1+q^2)\cdots(1+q^n)$$

$$= \sum_{n=0}^{\infty} S(n) q^n$$

$T(m) =$ # of inequivalent solns of

$(*)$ $u^2 - 6v^2 = m$ with $u + 3v \equiv \pm 1 \; (12)$

over those with $u + 3v \equiv \pm 5 \; (12)$

How do we go about finding an exact formula for $T(m)$?

Extend wt system from solns to $(*)$ to all of $\alpha = u + \sqrt{6}v \in \mathbb{Z}[\sqrt{6}]$

$$f(\alpha) = \begin{cases} +1 \text{ if either } v \text{ even}, u+3v \equiv \pm 1 \; (12) \\ \quad \text{or } v \text{ odd}, 2u+3v \equiv \pm 1 \; (12) \\ -1 \text{ if either } v \text{ even}, u+3v \equiv \pm 5 \; (12) \\ \quad \text{or } v \text{ odd}, 2u+3v \equiv \pm 5 \; (12) \end{cases}$$

# Proof of ADH Theorem

ADH developed $f$ & its relationship with $T$ until they obtained an exact formula for $T(m)$

For example.

- $R$ is $\alpha \in \mathbb{Z}[\sqrt{6}]$, $N(\alpha)$ relatively prime to 6

  $f$ is multiplicative. If $\alpha, \beta \in R$

  then $f(\alpha\beta) = f(\alpha)f(\beta)$

- If $\alpha, \alpha'$ are associates, then $f(\alpha) = f(\alpha')$
- If $m \equiv 1 \ (24)$ $\quad T(m) = \sum_{N(\alpha)=m} f(\alpha)$

where the sum is over nonassociate $\alpha$'s with norm $N(\alpha) = m$

- If $m \equiv 1 \ (6)$, $m \not\equiv 1 \ (24)$ then $T(m) = 0$
- $T$ is multiplicative. If $m, n$ relatively prime and $m \equiv n \equiv 1 \ (6)$ then

  $T(mn) = T(m)T(n)$

# Proof of ADH Theorem

ADH developed $f$ & its relationship with $T$

in order to obtain an exact formula for $T(m)$

— o —

$f(\alpha\beta) = f(\alpha)f(\beta) \quad \alpha, \beta \in R \subseteq \mathbb{Z}[\sqrt{6}]$

Proof involves considering cases

$\alpha = u + \sqrt{6}v, \ \beta = r + \sqrt{6}s$

$\alpha\beta = x + \sqrt{6}y, \ x = ur + vs, \ y = us + vr$

$u, r$ odd

three cases for parity of $v, s$

case 1: both $v, s$ even

gives $y$ even

so in def'n of $f$ we only need to consider

$x + 3y \mod 12$

$(x + 3y) = (u + 3v)(r + 3s) - 3vs$

$\equiv (u + 3v)(r + 3s) \mod 12$

consider cases $u + 3v, r + 3s \mod 12$ ✓

# Proof of ADH Theorem

## An exact formula for $T(m)$

**Theorem** (ADH) Let $m \neq 1$ be an integer with $m \equiv 1 \pmod 6$. Suppose $m = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r}$. Then

$$T(m) = T(p_1^{e_1})\, T(p_2^{e_2}) \cdots T(p_r^{e_r})$$

where

$$T(p^e) = \begin{cases} 0 & \text{if } p \not\equiv 1 \pmod{24},\ e \text{ odd} \\ 1 & \text{if } p \equiv 13, 19 \pmod{24},\ e \text{ even} \\ (-1)^{e/2} & \text{if } p \equiv 7 \pmod{24},\ e \text{ even} \\ e+1 & \text{if } p \equiv 1 \pmod{24}\ T(p) = 2 \\ (-1)^e (e+1) & \text{if } p \equiv 1 \pmod{24}\ T(p) = -2 \end{cases}$$

In particular, $T(m) = 0$ iff there is some $i$ for which $p_i \not\equiv 1\ (24)$ and $e_i$ is odd

Examples

$S(n) = T(24n+1)$

$$G(q) = \sum_{n=0}^{\infty} S(n) q^n$$

$$= 1 + q - q^2 + 2q^3 + \cdots$$

$$-1 = S(2) = T(24 \cdot 2 + 1)$$

$$= T(49)$$

$$= T(7^2) \qquad p \equiv 7 \ (24), \ e = 2 \text{ even}$$

$$= (-1)^{e/2} \qquad e = 2$$

$$= -1.$$

# Proof of ADH Theorem

Thm (ADH) Let $\sigma(q) = \sum_{n=0}^{\infty} S(n) q^n$

$S(n)$ is almost always zero, that is the set of $n$ for which $S(n) \neq 0$ has density zero

On the other hand, $S(n)$ takes on every integer value infinitely often.

Proof of first part

Given $x > 0$

Let $P_x = \{ p \text{ prime} \mid p \leq x, p \equiv 7 \pmod{24} \}$

If $S(n) \neq 0$ and $p \in P_x$ then

exponent of $p$ in $24n+1$ is even

$p \equiv 7 \ (24) \Rightarrow \left(\frac{6}{p}\right) = -1$

For each $p \in P_x$ either $p \nmid 24n+1$ or $p^2 \mid 24n+1$

Proof of AOW part one

The density of such primes is

$$\prod_{p \in P_x} \Big( \underbrace{1 - \frac{1}{p}}_{p \nmid 24n+1} + \underbrace{\frac{1}{p^2}}_{p^2 \mid 24n+1} \Big)$$

$$< \prod_{p \in P_x} \left( 1 - \frac{1}{2p} \right) \quad , \quad \frac{1}{p^2} < \frac{1}{2p}$$

$$< \prod_{p \in P_x} e^{-1/2p} = \exp\left( -\frac{1}{2} \sum_{p \in P_x} \frac{1}{p} \right)$$

but $\displaystyle\sum_{p \equiv 7 \pmod{24}} \frac{1}{p}$ diverges by strong Dirichlet Theorem

so as $x \to \infty$ we see the density of the set $n$, $S(n) \neq 0$ is zero

# Proof of ADH Theorem

Thm (ADH) Let $\sigma(q) = \sum_{n=0}^{\infty} S(n) q^n$

$S(n)$ is almost always zero, that is the set of $n$ for which $S(n) \neq 0$ has density zero

On the other hand, $S(n)$ takes on every integer value infinitely often.

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Proof of second part

Given an integer $k$

If $k = 0$ done by first part

Suppose $k \neq 0$

Recall $S(3) = 2$

So $2 = S(3) = T(24 \cdot 3 + 1) = T(73)$

# Proof of ADH part II

Suppose $k > 0$, $p \equiv 13$ or $19 \bmod 24$

$$\delta\left(\frac{73^{k-1}p^2 - 1}{24}\right) = \tau(73^{k-1}p^2)$$

$$= \tau(73^{k-1})\,\tau(p^2) \qquad \tau \text{ mult.}$$

$$= (k-1+1) \cdot 1$$

$$= k$$

$\tau(p^2) = 1$ because $p \equiv 13$ or $19 \pmod{24}$

$\tau(73^{k-1}) = k$ because $p \equiv 1 \ (24)$

inf many primes of form $p \equiv 13 \pmod{24}$

means we can hit $k$ inf many times

Other $q$-series similar to $\sigma(q)$

Andrews ; Carson, Favero, Lessinger
Zubairy

$$\sum_{n=0}^{\infty} \Delta(n) q^n = \sum_{n=0}^{\infty} q^{n(n+1)} (1+q^{2n+1})_n^{\times 2} \cdot \sum_{j=-n} q^{-j^2}$$

- $\Delta(n)$ counts inequivalent solns to $8n+1 = x^2 - 2y^2$
- $\Delta(n)$ is almost always equal to zero
- $\Delta(n)$ is equal to any given positive integer inf. often

$d$? $\boxed{-1, -2, 2}$ Kaplansky's?

6 $\quad$ 2, 3, 6

## Summary

Two Problems

- Kaplansky's Theorem on the simultaneous representations of primes by binary quadratic forms
- A $q$-series from Ramanujan's lost Notebook & two Conjectures of Andrews

Underlying themes & questions

- Threefield identities
- Relating the Fourier coefficients of a $q$-series to quadratic forms
- Two types of Hecke-type double sums

$$\Big(\sum_{r,s\geq 0} \pm \sum_{r,s<0}\Big)(-1)^{r+s} x^r y^s q^{a\binom{r}{2}+brs+c\binom{s}{2}}$$

Next time

- intro

  modular forms

  mock modular forms

  quantum modular forms