Lectures 4 & 5

Topics:

- Simultaneous representations of primes by binary quadratic forms
  a) Kaplansky
- A q-series from Ramanujan's Lost Notebook and two papers
  a) Andrews, Dyson, Hickerson
  b) Cohen
- An intro to quantum modular forms after Zagier

Notation $\qquad q \in \mathbb{C} \quad 0<|q|<1$

$$(x)_\infty = (x;q)_\infty = \prod_{i=0}^{\infty} (1-q^i x)$$

$$j(z;q) = \sum_{n=-\infty}^{\infty} (-1)^n z^n q^{\binom{n}{2}}$$

$$\text{JTP id} \quad = (z;q)_\infty (q/z;q)_\infty (q;q)_\infty$$

$$J_{a,m} := j(q^a; q^m)$$

$$\overline{J}_{a,m} := j(-q^a; q^m)$$

$$J_m := J_{m,3m} = \prod_{i=1}^{\infty} (1-q^{im})$$

partial theta fn $\displaystyle \sum_{n=0}^{\infty} (-1)^n z^n q^{\binom{n}{2}}$

false theta fn $\displaystyle \sum_{n=-\infty}^{\infty} sg(n) (-1)^n z^n q^{\binom{n}{2}}$

$$sg(n) := \begin{cases} 1 & n \geq 0 \\ -1 & n < 0 \end{cases}$$

Underlying themes & questions

- Three field identities

  Gauss
  Serre
  Andrews, Dyson, Hickerson
  Cohen

- You have a weight-one q-series. How do you go about relating the Fourier Coefficients $c(n)$ in

$$\sum_{n=0}^{\infty} c(n) q^n$$

  to the sum of weights over representations of $an+b = x^2 - dy^2$?

  How do you find a, b, d?

  How do you attach weights to the solutions?

Underlying themes & questions

For example (RLN)

$$\sigma(q) := 1 + \sum_{n=1}^{\infty} q^{n(n+1)/2} \Big/ (1+q)(1+q^2)\cdots(1+q^n)$$

$$= \sum_{n=0}^{\infty} S(n)\, q^n$$

$$= 1 + q - q^2 + 2q^3 + \cdots + 4q^{45} + \cdots + 6q^{1609} + \cdots$$

How do you obtain

$$\sigma(q) = \sum_{n=0}^{\infty} (-1)^n q^{n(3n+1)/2} (1 - q^{2n+1}) \sum_{j=-n}^{n} (-1)^j q^{-j^2} \;?$$

How do you know

$S(n)$ is the excess of the number of inequivalent solutions of

$$24n+1 = u^2 - 6v^2$$

with $u + 3v \equiv \pm 1 \pmod{12}$ over the number of them with $u + 3v \equiv \pm 5 \pmod{12}$?

Underlying themes & questions

When can we write a q-series as

(A) $$\left(\sum_{r,s\geq 0} - \sum_{r,s<0}\right)(-1)^{r+s}x^r y^s q^{a\binom{r}{2}+brs+c\binom{s}{2}}$$

(B) $$\left(\sum_{r,s\geq 0} + \sum_{r,s<0}\right)(-1)^{r+s}x^r y^s q^{a\binom{r}{2}+brs+c\binom{s}{2}}$$

(A) theta functions

mock theta functions

(B) false theta functions

RLN's $\sigma(q)$ (Andrews, Dyson, Hickerson)

(Cohen)

Q: What are the building block for (B)?

Underlying themes & questions

(A) Hecke-type double-sums

$$f_{a,b,c}(x,y,q) := \left(\sum_{r,s\geq 0} - \sum_{r,s<0}\right)(-1)^{r+s}x^r y^s q^{a\binom{r}{2}+brs+c\binom{s}{2}}$$

(B)

$$g_{a,b,c}(x,y,q) := \left(\sum_{r,s\geq 0} + \sum_{r,s<0}\right)(-1)^{r+s}x^r y^s q^{a\binom{r}{2}+brs+c\binom{s}{2}}$$

$$f_{1,2,1}(q,q,q) = \overline{J}_1^2 = \prod_{n=1}^{\infty}(1-q^n)^2$$

$$f_{1,2,1}(q,-q,q) = 2\overline{J}_{1,4}\, m(q,q^3,-1)$$

$$g_{1,2,2}(q,-q^3,q) = \sum_{n=-\infty}^{\infty} sg(n)(-1)^n q^{n(n+1)/2}$$

$$g_{1,5,1}(-q,-q,q) - q^2 g_{1,5,1}(-q^4,-q^4,q) = \sigma(q)$$

$$sg(n) := \begin{cases} 1 & n\geq 0 \\ -1 & n<0 \end{cases}$$

Simultaneous Representations of primes by binary quadratic forms

Theorem (Kaplansky)

A prime p, $p \equiv 1 \pmod{16}$ is representable by both or none of the quadratic forms $x^2 + 32y^2$ and $x^2 + 64y^2$

A prime p, $p \equiv 9 \pmod{16}$ is representable by exactly one of the quadratic forms

Examples

$\overline{p \equiv 1 \pmod{16}}$

$p = 17, 97$ neither

$p = 113 = 9^2 + 32 \cdot 1^2 = 7^2 + 64 \cdot 1^2$

$p \equiv 9 \pmod{16}$

$p = 41 = 3^2 + 32 \cdot 1^2$ $=$ no

$p = 73 =$ no $= 3^2 + 64 \cdot 1^2$

$p = 89 =$ no $= 5^2 + 64 \cdot 1^2$

Simultaneous Representations of primes by binary quadratic forms

Theorem (Kaplansky)

A prime p, $p \equiv 1 \pmod{16}$ is representable by both or none of the quadratic forms $x^2 + 32y^2$ and $x^2 + 16y^2$

A prime p, $p \equiv 9 \pmod{16}$ is representable by exactly one of the quadratic forms

—o—

Python Code!

Check case

$p \equiv 1 \pmod{16}$, p prime, $p < 10{,}000$

$p \equiv 9 \pmod{16}$, p prime, $p < 10{,}1000$

A q-series from the Lost Notebook

In the unsolved problem section of the American Mathematical Monthly (Nov 1986) George Andrews stated the following q-series found in Ramanujan's Lost Notebook.

$$\sigma(q) = 1 + \sum_{n=1}^{\infty} q^{n(n+1)/2} \Big/ (1+q)(1+q^2)\cdots(1+q^n)$$

$$= \sum_{n=0}^{\infty} S(n) q^n$$

$$= 1 + q - q^2 + 2q^3 + \cdots + 4q^{45} + \cdots$$

A q-series from the Lost Notebook

$$\sigma(q) = 1 + \sum_{n=1}^{\infty} q^{n(n+1)/2} \Big/ (1+q)(1+q^2)\cdots(1+q^n)$$

$$= \sum_{n=0}^{\infty} S(n) q^n$$

$$= 1 + q - q^2 + 2q^3 + \cdots + 4q^{45} + \cdots + 6q^{1609} + \cdots$$

Andrews observed that the coefficients are slow to grow, for example no coeff of q^n for $n \le 1600$ is greater than 4 in absolute value and he observed that the majority of $S(n)$ equal zero

Table: fraction of $n \le j$ with $S(n) = 0$

j	100	200	...	1000
fraction	0.44	0.475		0.562

A q-series from the Lost Notebook

$$\sigma(q) = 1 + \sum_{n=1}^{\infty} q^{n(n+1)/2} \Big/ (1+q)(1+q^2)\cdots(1+q^n)$$

$$= \sum_{n=0}^{\infty} S(n) q^n$$

$$= 1 + q - q^2 + 2q^3 + \cdots + 4q^{45} + \cdots + 6q^{1609} + \cdots$$

Python code!

- list of coefficients $c(n)$, $n < 10{,}000$

- $\forall m = 1, 2, 3, \ldots$

  what is first n where $|S(n)| = m$?

- how does the following grow

$$\frac{\#\{n \leq N \mid S(n) = 0\}}{N}$$

as $N \to \infty$?

A q-series from the Lost Notebook

$$\sigma(q) = 1 + \sum_{n=1}^{\infty} q^{n(n+1)/2} \Big/ (1+q)(1+q^2)\cdots(1+q^n)$$

$$= \sum_{n=0}^{\infty} S(n)$$

$$= 1 + q - q^2 + 2q^3 + \cdots + 4q^{45} + \cdots + 6q^{1609} + \cdots$$

Andrews made the following two conjectures

Conjecture 1: $\lim_{n \to \infty} \sup |S(n)| = +\infty$

Conjecture 2: $S(n) = 0$ inf. many n.

Three people wrote back:

Dean Hickerson

Freeman Dyson

Henri Cohen

A q-series from the Lost Notebook

There are actually two functions in play here

$$\sigma(q) = \sum_{n=0}^{\infty} q^{n(n+1)/2} \Big/ (1+q)(1+q^2)\cdots(1+q^n)$$

$$\sigma^*(q) = 2\sum_{n=1}^{\infty} (-1)^n q^{n^2} \Big/ (1-q)(1-q^3)\cdots(1-q^{2n-1})$$

Dyson:

This pair of functions σ & σ^* is today an isolated curiosity. But I am convinced that like so many other beautiful things in Ramanujan's Garden, it will turn out to be a special case of a broader mathematical structure ... within which the mock theta functions will also find a place ...

Three field identities (Notation)

—o—

Legendre Symbol

Let p be an odd prime number

$$\left(\frac{a}{p}\right) := \begin{cases} 1 & \text{if } a \text{ is a quadratic res mod } p \\ -1 & \text{if } a \text{ is a nonresidue mod } p \\ 0 & \text{if } a \equiv 0 \pmod{p} \end{cases}$$

Examples

a) $$\left(\frac{-1}{p}\right) = (-1)^{\frac{p-1}{2}} = \begin{cases} 1 & \text{if } p \equiv 1 \pmod{4} \\ -1 & \text{if } p \equiv 3 \pmod{4} \end{cases}$$

b) $$\left(\frac{2}{p}\right) = (-1)^{\frac{p^2-1}{8}} = \begin{cases} 1 & \text{if } p \equiv 1, 7 \pmod{8} \\ -1 & \text{if } p \equiv 3, 5 \pmod{8} \end{cases}$$

c) Fermat's two square theorem

$$p = x^2 + y^2 \iff p \equiv 1 \mod 4 \left[\text{or } \left(\frac{-1}{p}\right) = 1\right]$$

Three field identities (Notation)

—o—

Jacobi Symbol

- generalization of the Legendre Symbol
- For any integer a and any positive integer n,

  $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$

  The Jacobi Symbol is defined as the product of the Legendre Symbols

$$\left(\frac{a}{n}\right) := \left(\frac{a}{p_1}\right)^{\alpha_1} \left(\frac{a}{p_2}\right)^{\alpha_2} \cdots \left(\frac{a}{p_k}\right)^{\alpha_k}$$

Threefield identities

Two types of threefield identities

Let D & E be distinct square-free integers not equal to 1 and let F be the square-free part of DE.

There is an identity between representations of odd integers n for which the Jacobi Symbols

$$\left(\frac{D}{n}\right) = \left(\frac{E}{n}\right) = \left(\frac{F}{n}\right) = 1$$

by quadratic forms associated with the fields

$$\mathbb{Q}(\sqrt{D}),\ \mathbb{Q}(\sqrt{E}),\ \mathbb{Q}(\sqrt{F})$$

Three field identities

—o—

Type I

The case D, E, F are all positive

example

$\sigma(q)$ $D = 2, E = 3, F = 6$ (later)

Type II

The case two of the integers say D, E are negative and one say F is positive

example:

Kaplansky's Theorem (today)

Threefield identities

Notation

$$(x)_\infty = (x;q)_\infty = \prod_{i=0}^{\infty} (1-q^i x)$$

$$j(x;q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{\binom{n}{2}} x^n$$

$$= (x;q)_\infty (q/x;q)_\infty (q;q)_\infty$$

$$J_{a,m} := j(q^a;q^m)$$

$$\overline{J}_{a,m} := j(-q^a;q^m)$$

$$J_m := J_{m,3m} = \prod_{i=1}^{\infty} (1-q^{mi})$$

$$f_{a,b,c}(x,y,q)$$

$$:= \left(\sum_{r,s\geq 0} - \sum_{r,s<0}\right) (-1)^{r+s} x^r y^s q^{a\binom{r}{2}+brs+c\binom{s}{2}}$$

Three field identities

—o—

Type $\underline{\text{II}}$ (in more depth)

Here the generating functions turn out to be theta functions and the identity is two theta functions expressed in terms of J's and a Hecke-type double-sum whose weight system depends on the region being summed over

Ex $D=-1,\ E=-2,\ F=2$

$$J_{1,4}J_{2,4} = J_{1,2}\overline{J}_{1,4}$$
$$= f_{1,3,1}\left(q^{3/4}, -q^{3/4}, -q^{1/2}\right)$$

Application: Kaplausky's Theorem!

Lecture so far (Recap)

- introduced two sets of results
  - Kaplansky's Theorem
  - A q-series from RLN &
    conjectures of Andrews
- underlying themes & questions
  - three field identities
  - relating the Fourier coeffients of a q-series $\sum_{n=0}^{\infty} c(n) q^n$ to the solutions of a quadratic form $an+b=x^2-dy^2$
  - Hecke-type double-sums
    - two types of symmetry
    - building blocks

Simultaneous Representations of primes by Binary quadratic forms

- Kaplansky's Theorem
- understanding q-series

Steps:

- A related three field identity
- Prove the three field identity
- Three field identity in terms of weighted solution sets to quadratic forms
- Prove Kaplansky's Theorem using weighted solution sets
- Confirm that the weighted solution sets give the generating functions found in the three field identity.

Kaplansky's Theorem

Theorem (Kaplansky)

A prime p, $p \equiv 1 \pmod{16}$ is representable by both or none of the quadratic forms

$x^2 + 32y^2$ and $x^2 + 64y^2$

A prime p, $p \equiv 9 \pmod{16}$ is representable by exactly one of the quadratic forms.

<u>Proof</u> (Kaplansky)

Used two well-known results

a) Gauss

2 is a 4^{th} power modulo prime p

$\iff$ p is representable by $x^2 + 64y^2$

b) Barrucand and Cohn

-4 is an 8th power modulo prime $p \iff p$ is represtable by $x^2 + 32y^2$

Kaplansky's Theorem

Notation

$$(x)_\infty = (x;q)_\infty = \prod_{i=0}^{\infty} (1-q^i x)$$

$$j(x;q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{\binom{n}{2}} x^n$$

$$= (x;q)_\infty (q/x;q)_\infty (q;q)_\infty$$

$$J_{a,m} := j(q^a; q^m)$$

$$\overline{J}_{a,m} := j(-q^a; q^m)$$

$$J_m := J_{m,3m} = \prod_{i=1}^{\infty} (1-q^{mi})$$

$$f_{a,b,c}(x,y,q)$$

$$:= \left(\sum_{r,s\geq 0} - \sum_{r,s<0}\right) (-1)^{r+s} x^r y^s q^{a\binom{r}{2}+brs+c\binom{s}{2}}$$

Proof of Kaplansky's Theorem

preliminaries

Three field

$$J_{1,4}\bar{J}_{2,4}=J_{1,2}\bar{J}_{1,4}$$

$$=f_{1,3,1}\left(q_\chi^{3/4}, -q_\chi^{3/4}, -q_\chi^{1/2}\right)$$

$D=-1$, $E=-2$, $F=2$

identity between representations of odd integers n for which the Jacobi Symbols

$$\left(\frac{D}{n}\right)=\left(\frac{E}{n}\right)=\left(\frac{F}{n}\right)=1$$

by quadratic forms associated with the fields

$\mathbb{Q}(\sqrt{D})$, $\mathbb{Q}(\sqrt{E})$, $\mathbb{Q}(\sqrt{F})$

Proof of Kaplansky's Theorem

preliminaries

$$J_{1,4}J_{2,4} = J_{1,2}J_{1,4} = f_{1,3,1}(q^{3/4}, -q^{3/4}, -q^{1/2})$$

$$= 1 - q - 2q^2 + q^3 + 2q^5 + q^6 - 2q^9 + \cdots$$

$$=: \sum_{n=0}^{\infty} c(n) q^n$$

We can relate $c(n)$'s to the solution sets of three different quadratic forms:

$8n+1 = x^2 + y^2 \qquad d = -1$

$8n+1 = x^2 + 2y^2 \qquad d = -2$

$8n+1 = x^2 - 2y^2 \qquad d = 2$

general form $an+b = x^2 - dy^2$

Proof of Kaplansky's Theorem

Proof of three field identity

$$J_{1,4}J_{2,4} = J_{1,2}\overline{J}_{1,4} = f_{1,3,1}(q^{3/4}_x, -q^{3/4}_x, -q^{1/2}_x)$$

First equality follows from a product rearrangement

Second equality follows from an $f_{a,b,c}(x,y,q)$ expansion

Proof of Kaplansky's Theorem

Proof of threefield identity

Proof of first equality

$$J_{1,4}\overline{J}_{2,4} = J_{1,2}\overline{J}_{1,4}$$

Notation $(x;q)_\infty = \prod_{i=0}^{\infty} (1-q^i x)$

$$J_{a,m} = j(q^a ; q^m)$$

$$\overline{J}_{a,m} = j(-q^a ; q^m)$$

Jacobi Triple Product Id.

$$j(x;q) = (x;q)_\infty (q/x;q)_\infty (q;q)_\infty$$

Examples of Product Rearrangements

$$(q;q)_\infty = (1-q)(1-q^2)(1-q^3)(1-q^4)(1-q^5)\cdots$$

$$= (1-q)(1-q^3)(1-q^5)\cdots(1-q^2)(1-q^4)\cdots$$

$$= (q;q^2)_\infty \, (q^2;q^2)_\infty$$

odd power even powers

Proof of Kaplansky's Theorem

Proof of three field identity

Examples of Product Rearrangements:

Ex: $(q^2;q^4)_\infty = (1-q^2)(1-q^6)(1-q^{10})(1-q^{14})\cdots$

$$= (1-q)(1+q)(1-q^3)(1+q^3)(1-q^5)(1+q^5)\cdots$$

$$= (1-q)(1-q^3)(1-q^5)\cdots(1+q)(1+q^3)(1+q^5)\cdots$$

$$= (q;q^2)_\infty(-q;q^2)_\infty$$

In general

$$(q^{2a};q^{2m})_\infty = (q^a;q^m)_\infty(-q^a;q^m)_\infty$$

above is $a=1$, $m=2$

Proof of Kaplansky's Theorem

Proof of three field identity

First equality $\quad J_{1,4}\overline{J}_{2,4} = \overline{J}_{1,2}\overline{J}_{1,4}$

$J_{1,4}\overline{J}_{2,4}$

$=$ use Jacobi Triple Product Identity

$$= (q;q^4)_\infty (q^3;q^4)_\infty (q^4;q^4)_\infty (q^2;q^4)^2_\infty (q^4;q^4)_\infty$$

$$= (q;q^2)_\infty (q^4;q^4)_\infty (q^2;q^4)^2_\infty (q^4;q^4)_\infty$$

$$= (q;q^2)_\infty (q^2;q^2)_\infty (q^2;q^4)_\infty (q^4;q^4)_\infty$$

$$= (q;q^2)^2_\infty (q^2;q^2)_\infty (-q;q^2)_\infty (q^4;q^4)_\infty$$

$$= J_{1,2}\,(-q;q^4)_\infty (-q^3;q^4)_\infty (q^4;q^4)_\infty$$

$$= J_{1,2}\,\overline{J}_{1,4}$$

Proof of Kaplansky's Theorem

Proof of threefield identity

Notation

$$(x)_\infty = (x;q)_\infty = \prod_{i=0}^{\infty} (1-q^i x)$$

$$j(z;q) = \sum_{n=-\infty}^{\infty} (-1)^n z^n q^{\binom{n}{2}}$$

$$= (z;q)_\infty (q/z;q)_\infty (q;q)_\infty$$

$$J_{a,m} := j(q^a;q^m)$$

$$\overline{J}_{a,m} := j(-q^a;q^m)$$

$$J_m := J_{m,3m} = \prod_{i=1}^{\infty} (1-q^{im})$$

$$m(x,q,z) := \frac{1}{j(z;q)} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{\binom{n}{2}} z^n}{1-q^{n-1} xz}$$

Proof of Kaplansky's Theorem

Proof of threefield identity

Proof of second equality (sketch)

$$J_{1,2}\overline{J}_{1,4} = f_{1,3,1}(q^{3/4}, -q^{3/4}, -q^{1/2})$$

$$f_{1,3,1}(x,y,q) = j(y;q)\,m(-q^5x/y^3, q^8, y/x)$$

$$+ j(x;q)\,m(-q^5y/x^3, q^8, x/y)$$

$$-qxy\,\frac{J_{2,4}J_{8,16}\,j(q^3xy;q^8)\,j(q^{16}x^2y^2;q^{16})}{j(-q^3x^2;q^8)\,j(-q^3y^2;q^8)}$$

$$f_{1,3,1}(q^{3/4}, -q^{3/4}, -q^{1/2}) = \cdots$$

$$= \cdots = J_{1,2}\overline{J}_{1,4}$$

Proof of Kaplansky's Theorem

The first equality $\bar{J}_{1,4}\bar{J}_{2,4} = \bar{J}_{1,2}\bar{J}_{1,4}$ gives Kaplansky's Thm

The first equality gives a relationship between solutions to

$$8k+1 = x^2+y^2$$

and solutions to

$$8k+1 = x^2+2y^2$$

Proof of Kaplansky's Theorem

$$J_{1,4}J_{2,4} = \overline{J}_{1,2}\overline{J}_{1,4} = \sum_{n=0}^{\infty} c(n) q^{n}$$

First equality says

$8k+1 = x^2+y^2$: Fix $x>0$. The coeff $c(k)$ is the excess of the number of inequivalent solns with

$x \equiv \pm 1\ (8)$, $y \equiv 0\ (8)$ or $x \equiv \pm 3\ (8)$ $y \equiv 4\ (8)$

over the number of inequivalent solns with

$x \equiv \pm 3\ (8)$ $y \equiv 0\ (8)$ or $x \equiv \pm 1\ (8)$ $y \equiv 4\ (8)$.

$8k+1 = x^2+2y^2$ Here x odd y even. The coeff $c(k)$ is the number of inequivalent solns with

$y \equiv 0\ (4)$

over the number with

$y \equiv 2\ (4)$

Proof of Kaplansky's Theorem

Thm (Kaplansky)

A prime p, $p \equiv 1 \pmod{16}$ is representable by both or none of the quadratic forms

$x^2 + 32y^2$ and $x^2 + 64y^2$

A prime p, $p \equiv 9 \pmod{16}$ is representable by exactly one of the quadratic forms

Proof follows from previous relationship.

question: how?

bigger question:

In $an + b = x^2 - dy^2$, how do we find a, b, d, and how do we find the weights?

Proof of Kaplansky's Theorem

With a simple change of variables we can rewrite the weights of solutions of $8k+1 = x^2+y^2$ and $8k+1 = x^2+2y^2$ in terms of the weights of solutions of $8k+1 = x^2+16y^2$ and $8k+1 = x^2+8y^2$

We can rewrite the weighted soln sets accordingly

Proof of Kaplansky's Theorem

The excess of the number of inequivalent solns of $8k+1=x^2+16y^2$ $(x>0)$

with

$x\equiv\pm1\ (8)$, y even or $x\equiv\pm3\ (8)$ y odd

over

$x\equiv\pm3\ (8)$, y even or $x\equiv\pm1\ (8)$ y odd

equals

the number of excess solns of

$8k+1 = x^2+8y^2$ $(x>0)$

with

y even

over

y odd.

$$J_{1,4}J_{2,4}=J_{1,2}\overline{J}_{1,4}=:\sum_{n=0}^{\infty}c(n)q^n$$

Proof of Kaplansky's Theorem

So if $p = 8k+1$ is prime

there are exactly two representations by

each of these forms $(y \to -y)$

if $p \equiv 1 \pmod 8$ is prime, the p's

unique form $x^2 + 16y^2$ $(x > 0)$

has $x \equiv \pm 1\ (8)$ y even or $x \equiv \pm 3\ (8)$ y odd

iff p's unique representation $x^2 + 8y^2$

$(x, y > 0)$ has y even

iff p has a representation of the form

$x^2 + 32y^2$

Proof of Kaplansky's Theorem

I) p prime $p \equiv 1 \pmod{16}$ is representable by both or none of the quadratic forms $x^2 + 32y^2$ and $x^2 + 64y^2$

<u>Proof</u>

$p \equiv 1 \pmod{16}$

In the representation $p = x^2 + 16y^2$ we must have $x \equiv \pm 1 \pmod{8}$.

Thus p's representation in this form has y even $\Leftrightarrow$ p has a representation of the form $x^2 + 32y^2$

II) similar.

Big Questions remaining

1) If one knows the generating fns.

$\bar{J}_{1,4}\bar{J}_{2,4}$ or $J_{1,2}\bar{J}_{1,4}$

how does one find a, b, d in

$an+b = x^2 - dy^2$

and the weight system?

2) If one just has a q-series how does one find a, b, d? how does one find the weight system? how does one go from the weight system to the generating fns?

Basic Questions

1) We have generating fns.
How do we find a, b, d in
$an + b = x^2 - dy^2$??
How do we find the weight system?

—o—

For a theta function $f(q)$ (and many fns)
there is an associated fractional exponent
λ such that $q^{\lambda} f(q)$ is in some ways
simpler than $f(q)$. Modularity properties
are easier to state. The λ for both

$J_{e,m} = j(q^e ; q^m)$ and $\bar{J}_{e,m} = j(-q^e ; q^m)$

is $\lambda = (m-2e)^2 / 8m$ " $=$ " a/b

Basic Questions

1) We can find a, b. How do we find d?

In $\overline{J}_{e_1, m_1} \circ \overline{J}_{e_2, m_2}$

(or with $\overline{J}$'s.)

$d = -$ square-free part of $m_1 \cdot m_2$

Examples

$J_{1,4} \overline{J}_{2,4}$

$$\lambda = \frac{(4-2\cdot 1)^2}{8\cdot 4} + \frac{(4-2\cdot 2)^2}{8\cdot 4} = \frac{1}{8} \text{ "=" } \frac{a}{b}$$

$d = -(\text{square-free } 4\cdot 4) = -1$

$8n+1 = x^2 + y^2$

what about weights?

Basic Questions

1) Examples

$J_{1,2}\,\overline{J}_{1,4}$

$$\lambda = \frac{(2-2\cdot 1)^2}{8\cdot 2} + \frac{(4-2\cdot 1)^2}{8\cdot 4} = \frac{1}{8} \text{"} = \text{"} \frac{a}{b}$$

$d = -(\text{square-free part } 2\cdot 4) = -2$

$8n+1 = x^2+2y^2$

what about the weights?

Basic Questions

1) Examples

Python Code!

Should we assume a & b are in lowest terms? No

$J_2 J_{3,12}$ (actually $J_{2,6} J_{3,12}$)

$$\lambda = \frac{(6-2\cdot 2)^2}{8\cdot 6} + \frac{(12-3\cdot 2)^2}{8\cdot 12} = \frac{4}{8\cdot 6} + \frac{36}{8\cdot 12}$$

$$= \frac{1}{12} + \frac{3}{8} = \frac{11}{24}$$

$d = -$square-free part $6\cdot 12 = -2$

$24n+11 = x^2+2y^2$ has no solns

$48n+22 = x^2+2y^2$ does have solns

Basic Questions

1) We have the generating fns

how do we find a, b, d in $an+b = x^2 - dy^2$

how do we find weight system?

how do we show weight system gives the generating fns?

2) same questions but we only start with a weight-one q-series

Basic Questions

1) Given generating function $\overline{J}_{1,2}\overline{J}_{1,4} = \sum_{n=0}^{\infty} c(n) q^n$

we suspect the Fourier Coefficients

$c(n)$ are related to the solutions

of $\quad 8n+1 = x^2 + 2y^2 \qquad (d=-2)$

We suspect we are counting over

weighted solutions.

How do we find the weights?

Find n such that

$$8n+1 = p \text{ a prime number}$$

and $\quad c(n) \neq 0$

Basic Questions

1) $\sum_{n=0}^{\infty} c(n) = \overline{J}_{1,2}\,\overline{J}_{1,4}$

$$= 1 - q - 2q^2 + q^3 + 2q^5 + q^6 - 2q^9 + \cdots$$

Find n, $8n+1 = p$ prime, $c(n) \neq 0$

Note that these are primes p

where $\left(d/p\right) = 1$, $\left(\frac{-2}{p}\right) = 1$

Examples (only listing $x > 0$)

$n=2$, $c(2) = -2$ $\quad 8n+1 = 17$

soln's $(3,-2)$ $(3,+2)$ each -1

$n=5$ $c(5) = 2$ $\quad 8n+1 = 41$

soln's $(3,-4)$ $(3,4)$ each $+1$

$n=9$ $c(9) = -2$ $\quad 8n+1 = 73$

solns $(1,-6)$ $(1,6)$ each -1

$n=11$ $c(11) = -2$ $\quad 8n+1 = 89$

solns $(9,-2)$ $(9,2)$ each -1

Basic Questions

1) $$\sum_{n=0}^{\infty} c(n) = \overline{J}_{1,2}\,\overline{J}_{1,4}$$
$$= 1 - q - 2q^2 + q^3 + 2q^5 + q^6 - 2q^9 + \cdots$$

Find n, $8n+1 = p$ prime, $c(n) \neq 0$

Note that these are primes p where $\left(\frac{d}{p}\right) = 1$, $\left(\frac{-2}{p}\right) = 1$

Examples (only listing $x > 0$)

Python code !

What happens when we don't restrict ourselves to cases $an+b =$ prime, $c(n) \neq 0$?

Basic Questions

1) $\sum_{n=0}^{\infty} c(n) q^n = \overline{J}_{1,2} \overline{J}_{1,4}$
$= 1 - q - 2q^2 + q^3 + 2q^5 + q^6 - 2q^9 + \cdots$

building up weight system

go from weight system to theta fn

look for a pattern

- Find n, $8n+1 = p$ prime, $c(n) \neq 0$
  make a list of the solutions
  These are p, $(-2/p) = 1$

- For p, $(-2/p) = -1$
  we look for n, $8n+1 = p^2$, $c(n) \neq 0$

$n=3$ $\quad c(3) = 1 \quad 8n+1 = 25$

solns $(5,0)$ $\quad$ wt $\underline{1}$

$n=6$ $\quad c(6) = 1 \quad 8n+1 = 49$

solns $(7,0)$ $\quad$ wt $\underline{1}$

Basic Questions

1) $\sum_{n=0}^{\infty} c(n) q^n = \overline{J}_{1,2} \overline{J}_{1,4}$

Building up weight system

Going from weight system to theta fns.

$n=2$ $c(2)=-2$ $8n+1=17$

solns $(3, \pm 2)$ each wt -1

$n=5$ $c(5)=2$ $8n+1=41$

solns $(3, \pm 4)$ each wt $+1$

how does above relate to solns to

$n=87$ $c(87)=-4$ $8n+1=17 \cdot 41=697$

$\left.\begin{array}{l}(7,+18),(7,-18)\\(25,+16),(25,-16)\end{array}\right\}$ each weight -1

solns & weights multiply.

Basic Questions

1) $p = x^2 + 2y^2$ $\quad q = u^2 + 2v^2$
$17 =$ $\quad 41 =$

$p = (x - \sqrt{-2}\,y)(x + \sqrt{-2}\,y)$ $\quad x = 3,\ y = 2$

$q = (u - \sqrt{-2}\,v)(u + \sqrt{-2}\,v)$ $\quad u = 3,\ v = 4$

note we actually have $\pm x, \pm y, \pm u, \pm v, \ldots$

Consider permutations for pq

$$pq = \left[(x - \sqrt{-2}\,y)(u - \sqrt{-2}\,v)\right]$$

$$\cdot \left[(x + \sqrt{-2}\,y)(u + \sqrt{-2}\,v)\right]$$

$$= \left[xu - 2yv - \sqrt{-2}(xv + yu)\right]$$

$$\left[xu - 2yv + \sqrt{-2}(xv + yu)\right]$$

$$= \left[9 - 2 \cdot 8 - \sqrt{-2}\,(12 + 6)\right] \ldots$$

$$= \left(-7 - \sqrt{-2} \cdot 18\right) \ldots$$

Basic Questions

1) Back to list of n, where

$8n+1 = p$ a prime, $c(n) \neq 0$

$n=2$ $c(2) = -2$ $p=17$

solns $(3, \pm 2)$ each weight -1

$n=5$ $c(5) = 2$ $p=41$

solns $(3, \pm 4)$ each weight $+1$

$n=9$ $c(9) = -2$ $p=73$

solns $(1, \pm 6)$ each weight -1

$\vdots$

Pattern: Here $8n+1 = x^2 + 2y^2$

x is odd, y even. The coeff $c(n)$

is the excess of the number of inequivalent

solns of $8n+1 = x^2+2y^2$ with

$y \equiv 0 \mod 4$ over those with

$y \equiv 2 \mod 4$.

$an+b = x^2+2y^2$

Basic Questions

$n = (x^2+2y^2-b)/a$

1) How do we go from our guess of the weight system to our generating function

$$\sum_{n=0}^{\infty} c(n) q^n = \overline{J}_{1,2}\,\overline{J}_{1,4}$$

$c(n)$ is number of inequiv solns of $8n+1 = x^2+2y^2$ with $y \equiv 0 \pmod 4$ over those with $y \equiv 2 \pmod 4$

$$\sum_{n=0}^{\infty} c(n) q^n = \frac{1}{2}\sum_{r,s} q^{[(2r+1)^2+2(4s)^2-1]/8}$$

$$-\frac{1}{2}\sum_{r,s} q^{[(2r+1)^2+2(4s+2)^2-1]/8}$$

exponent is n.

Basic Questions

2)

$$\sum_{n=0}^{\infty} c(n) q^n = \frac{1}{2}\sum_{r,s} q^{[(2r+1)^2+2(4s)^2-1]/8}$$

$$- \frac{1}{2}\sum_{r,s} q^{[(2r+1)^2+2(4s+2)^2-1]/8}$$

$$= \frac{1}{2}\sum_{r,s} (-1)^s q^{[(2r+1)^2+2(2s)^2-1]/8}$$

$$= \frac{1}{2}\sum_{r,s} (-1)^s q^{r^2/2+r/2+s^2}$$

$$= \frac{1}{2}\sum_{r} q^{\binom{r}{2}+r} \cdot \sum_{s} (-1)^s q^s q^{2\binom{s}{2}}$$

$$= \frac{1}{2}\sum_{r} (-1)^r (-q)^r q^{\binom{r}{2}} \sum_{s} (-1)^s q^s q^{2\binom{s}{2}}$$

$$= \frac{1}{2}\overline{J}_{0,1} J_{1,2} = \overline{J}_{1,4} J_{1,2}$$ ✓

Summary

Two problems

- Kaplansky's Theorem & Simultaneous representations of primes by binary quadratic forms
- A q-series from RLN & two conjectures of Andrews

Underlying themes and questions

- Three field identities
- Relating Fourier coefficients of q-series to binary quadratic forms
- Two types of Hecke-type double-sums

Proof of Kaplansky's Theorem

Next time Lecture 5

- A q-series from the Lost Notebook

  $\sigma(q)$

  Andrews, Dyson, Hickerson

  Cohen

- intro to quantum modular forms

  (Zagier, Folsom)

- underlying themes & questions
  - threefield identities
  - relating Fourier coefficients of a q-series to the solutions of a quadratic form(s)
  - Hecke-type double-sums and their building blocks

Andrews, Dyson, Hickerson

Partitions and indefinite quadratic forms
Inv. Math 1988

C.F Gauss

Theorie der biquadratischen Reste, I

Cohen

q-identities and Maass Wave forms
Inv Math 1988

Kaplansky

The forms $x+32y^2$ and $x+64y^2$
Proc AMS 2003

Martenson

Threefield identities and simultaneous
representations of primes by quadratic forms
J Number Theory 2013

Serre

Modular forms of weight one and
Galois representations 1977

Andrews

Partitions with distinct evens 2009

Corson, Favero, Leisinger, Zubairy

Characters and q-series in $\mathbb{Q}(\sqrt{2})$

J. Number Theory 2004