The Circle Problem of Gauss, the Divisor Problem of Dirchlet, and Ramanujan's Interest in Them

September 17, 2020

**Our thoughts are with the millions of families who are suffering from economic difficulties,
serious illness,
and death due to covid-19**

Gauss's Circle Problem

The Dirichlet Divisor Problem

Two Related Identities from Ramanujan's Lost Notebook

Ramanujan's Passport Picture

Figure: Ramanujan

Quote from Ramanujan's First Letter to Hardy

Page 3, Item (4)

1, 2, 4, 5, 8, 9, 10, 13, 16, 17, 18, ... are numbers which are either themselves squares or which can be expressed as the sum of two squares.
The number of such numbers greater than A and less than B

$$= K \int_A^B \frac{dx}{\sqrt{\log x}} + \theta(x) \tag{1}$$

where $K = .764$ and $\theta(x)$ is very small when compared with the previous integral. K and $\theta(x)$ have been exactly found though complicated.

Remarks on (1)

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$$K = \sqrt{\frac{1}{2}\prod_{r}\left(\frac{1}{1-1/r^2}\right)},$$

where r runs through the primes of the form $4m+3$.

Quote from Hardy

The dominant term, viz. $KB(\log B)^{-1/2}$*, in Rammanujan's notation, was first obtained by Landau in 1908. Ramanujan had none of Landau's weapons at his command; ... It is sufficiently marvellous that he should have even dreamt of problems such as these, problems which it has taken the finest mathematicians in Europe a hundred years to solve ...*

G. H. Hardy
Collected Papers of Srinivasa Ramanujan, p. xxiv

G. H. Hardy

Quote from Watson

Quote from Watson

The most amazing thing about this formula is that it was discovered, apparently independently, by Ramanujan in his early days in India, and it appears in its appropriate place in his manuscript note-books.

G. N. Watson

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Appearances in Ramanujan's Notebooks

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- More space (the entire page) is devoted to this heuristic argument than to any other argument or proof in the notebooks.
- The third notebook may not have been available to Hardy and Watson. Watson's handwritten personal copy of the notebooks does not contain the third notebook.

Circle Problem

Let $r_2(n)$ denote the number of representations of the positive integer n as a sum of two squares. Different signs and different orders of the summands yield distinct representations. E.g.,

$$5 = (\pm 2)^2 + (\pm 1)^2, \qquad r_2(5) = 8.$$

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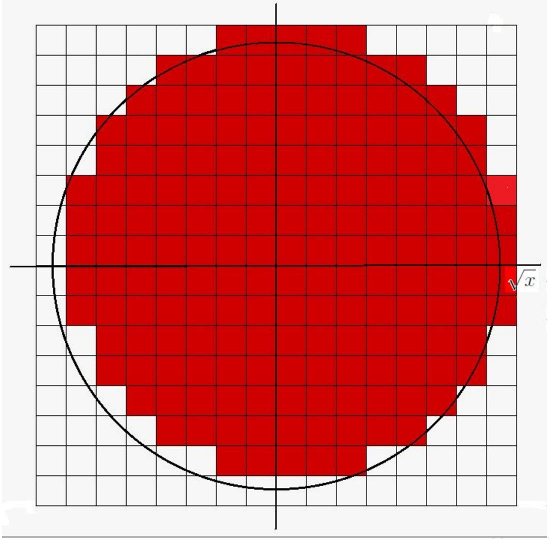
Each representation of n as a sum of two squares can be associated with a lattice point in the plane. For example, $5 = (-2)^2 + 1^2$ can be associated with the lattice point $(-2, 1)$. Then each lattice point can be associated with a unit square, say that unit square for which the lattice point is in the southwest corner.

Circle Problem



Circle Problem

$$R(x) := {\sum_{0 \le n \le x}}' r_2(n) = \pi x + P(x), \tag{2}$$

where the prime $'$ on the summation sign on the left side indicates that if x is an integer, only $\frac{1}{2} r_2(x)$ is counted.

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$$R(x) < \pi(\sqrt{x}+\sqrt{2})^2,$$
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$F(x) = O(G(x))$ as $x \to \infty$, if $\exists$ constants C and x_0 such that for $x \geq x_0$,

$$|F(x)| \leq C|G(x)|.$$

Circle Problem

Ramanujan (1914?) and Hardy (1915) proved that

$$\sum_{0\le n\le x}{}' r_2(n) = \pi x + \sum_{n=1}^{\infty} r_2(n)\left(\frac{x}{n}\right)^{1/2} J_1(2\pi\sqrt{nx}). \qquad (3)$$

$$J_\nu(z) := \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(\nu+n+1)}\left(\frac{z}{2}\right)^{\nu+2n}, \qquad 0<|z|<\infty, \qquad \nu\in\mathbb{C}.$$

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"The form of this equation was suggested to me by
Mr. S. Ramanujan, to whom I had communicated the analogous formula for $d(1)+d(2)+\cdots+d(n)$, where $d(n)$ is the number of divisors of n."

Sierpinski's Theorem and Huxley's Theorem

$$J_\nu(z) = \left(\frac{2}{\pi z}\right)^{1/2} \cos\left(z - \frac{1}{2}\pi\nu - \frac{1}{4}\pi\right) + O\left(\frac{1}{z^{3/2}}\right), \qquad |z| \to \infty.$$

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The current best result is due to M. N. Huxley in 2003, namely, for every $\epsilon > 0$,

$$P(x) = O(x^{131/416+\epsilon}), \tag{4}$$

as $x \to \infty$. Note that

$$\tfrac{131}{416} = 0.3149\ldots.$$

Another Beautiful Identity of Ramanujan as Recorded by Hardy

If $a, b > 0$, then

$$\sum_{n=0}^{\infty} \frac{r_2(n)}{\sqrt{n+a}} e^{-2\pi\sqrt{(n+a)b}} = \sum_{n=0}^{\infty} \frac{r_2(n)}{\sqrt{n+b}} e^{-2\pi\sqrt{(n+b)a}}, \qquad (5)$$

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If we differentiate (5) with respect to b, let $a \to 0$, replace $2\pi\sqrt{b}$ by s, and use analytic continuation, we find that, for Re $s > 0$,

$$\sum_{n=1}^{\infty} r_2(n) e^{-s\sqrt{n}} = \frac{2\pi}{s^2} - 1 + 2\pi s \sum_{n=1}^{\infty} \frac{r_2(n)}{(s^2 + 4\pi^2 n)^{3/2}},$$

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which was the key identity in Hardy's proof of

$$P(x) = \Omega_{\pm}(x^{1/4}), \qquad \text{as } x \to \infty.$$

Explanation of Notation

There exists a positive constant C_1 and a sequence $\{x_n\} \to \infty$ such that

$$P(x_n) > C_1 x_n^{1/4}, \quad n \geq 1.$$

There exists a positive constant C_2 and a sequence $\{x_n'\} \to \infty$ such that

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G. H. Hardy, On the expression of a number as the sum of two squares, Quart. J. Math. (Oxford) **46** (1915), 263–283.

An Elementary Formula

Identity of Jacobi

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$$r_2(n) = 4 \sum_{\substack{d|n \\ d \text{ odd}}} (-1)^{(d-1)/2}.$$

$$\begin{aligned}
\sum_{0<n\le x}{}' r_2(n) &= 4 \sum_{0<n\le x}{}' \sum_{d|n} \sin\left(\frac{\pi d}{2}\right) \\
&= 4 \sum_{0<dj\le x}{}' \sin\left(\frac{\pi d}{2}\right) \\
&= 4 \sum_{0<d\le x}{}' \left[\frac{x}{d}\right] \sin\left(\frac{\pi d}{2}\right),
\end{aligned}$$

where $[x]$ is the greatest integer $\le x$.

First Identity

An identity involving $r_2(n)$ found in a one-page manuscript published with Ramanujan's Lost Notebook.
p. 335

Ramanujan's First Claim

To state Ramanujan's claims, we need to first define

$$F(x) = \begin{cases} [x], & \text{if } x \text{ is not an integer,} \\ x - \frac{1}{2}, & \text{if } x \text{ is an integer,} \end{cases} \tag{6}$$

where, as customary, $[x]$ is the greatest integer less than or equal to x.

The First Entry

Entry

Let $F(x)$ be defined by (6), let $J_1(x)$ denote the ordinary Bessel function of order 1, let $0 < \theta < 1$, and let $x > 0$. Then

$$\sum_{n=1}^{\infty} F\left(\frac{x}{n}\right)\sin(2\pi n\theta) = \pi x\left(\frac{1}{2}-\theta\right) - \frac{1}{4}\cot(\pi\theta)$$
$$+\frac{1}{2}\sqrt{x}\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\left\{\frac{J_1\left(4\pi\sqrt{m(n+\theta)x}\right)}{\sqrt{m(n+\theta)}} - \frac{J_1\left(4\pi\sqrt{m(n+1-\theta)x}\right)}{\sqrt{m(n+1-\theta)}}\right\}.$$

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BCB, S. Kim and A. Zaharescu, *The circle and divisor problems, and double series of Bessel functions*, Adv. Math. **236** (2013), 24–59.

Sun Kim

Figure: Sun Kim at Graduation with Her Advisor

Theorem

If $0 < \theta < 1$, $x > 0$, *and* $J_1(x)$ *denotes the ordinary Bessel function of order* 1, *then*

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Version of BCB & Zaharescu

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BCB and A. Zaharescu, *Weighted divisor sums and Bessel function series*, Math. Ann. **335** (2006), 249–283.

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- Third Version: Interpret the double sum with $mn \to \infty$.
- Can we use the "extra" parameter θ to attack the *circle problem*?

The Special Case $\theta = \frac{1}{4}$

Corollary

For any $x > 0$,

$$R(x) = \sum_{0 \le n \le x}{}' r_2(n) = \pi x$$

$$+2\sqrt{x} \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \left\{ \frac{J_1\left(4\pi\sqrt{m(n+\frac{1}{4})x}\right)}{\sqrt{m(n+\frac{1}{4})}} - \frac{J_1\left(4\pi\sqrt{m(n+\frac{3}{4})x}\right)}{\sqrt{m(n+\frac{3}{4})}} \right\}.$$

Another Result from Ramanujan's First Letter to Hardy

(3) Let us take the number of divisors of natural numbers, viz. 1, 2, 2, 3, 2, 4, 2, 4, 3, 4, 2, ... (1 having 1 divisor, 2 having 2, 3 having 2, 4 having 3, 5 having 2, ...). The sum of such numbers to n terms

$$= n(2\gamma - 1 + \log n) + \frac{1}{2}\text{of the number of divisors of } n$$

where $\gamma = .5772156649\ldots$, the Eulerian Constant.

Dirichlet divisor problem

Let $d(n)$ denote the number of positive divisors of the positive integer n. Let

$$D(x) := {\sum_{n\le x}}' d(n),$$

where the prime $'$ indicates that if x is an integer, then we only count $\frac{1}{2}d(x)$. We see that

$$D(x) = \sum_{n\le x}^{prime} \sum_{d|n} 1 = {\sum_{dj\le x}}' 1 = {\sum_{d\le x}}' \sum_{1\le j\le x/d} 1 = {\sum_{d\le x}}' \left[\frac{x}{d}\right],$$

where $[x]$ is the greatest integer less than or equal to x.

Geometric Interpretation

If $n = dj$, as above, we observe that n is uniquely associated with the lattice point (d, j) in the first quadrant under or on the hyperbola $ab = x$. Hence, $D(x)$ is equal to the number of lattice points in the first quadrant under or on the hyperbola $ab = x$.

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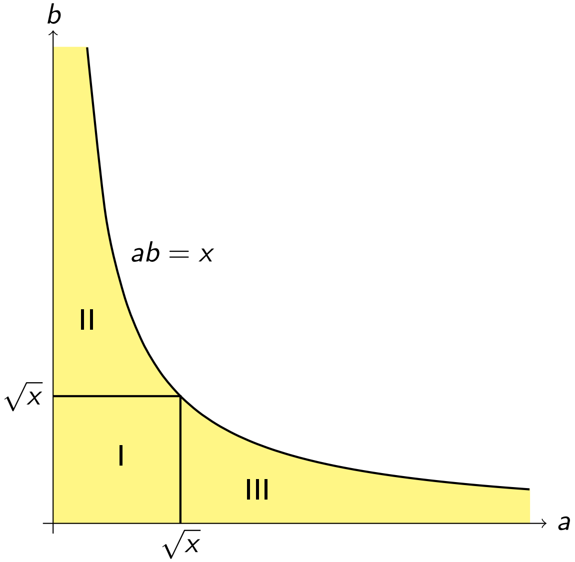
Dirichlet's divisor problem is equivalent to the problem of estimating the number of lattice points under or on a certain hyperbola.

Geometrical Interpretation



Dirichlet's Argument

We divide this region under the hyperbola $ab = x$ into three sections. Hence, if $\{x\}$ denotes the fractional part of x, as $x \to \infty$,

$$\begin{aligned} D(x) &= \sum_{d \le x} \left[\frac{x}{d}\right] = \sum_{d \le \sqrt{x}} \left[\frac{x}{d}\right] + \sum_{d \le \sqrt{x}} \left[\frac{x}{d}\right] - \left[\sqrt{x}\right]\left[\sqrt{x}\right] \\ &= 2 \sum_{d \le \sqrt{x}} \left(\frac{x}{d} + O(1)\right) - \left(\sqrt{x} - \{\sqrt{x}\}\right)^2 \\ &= 2x \sum_{d \le \sqrt{x}} \frac{1}{d} + O\left(\sqrt{x}\right) - x + O\left(\sqrt{x}\right) \\ &= 2x \left(\log \sqrt{x} + \gamma + O\left(1/\sqrt{x}\right)\right) - x + O\left(\sqrt{x}\right) \\ &= x \log x + (2\gamma - 1)x + O\left(\sqrt{x}\right), \end{aligned}$$

where in the penultimate step we used

$$\sum_{n \le x} \frac{1}{n} = \log x + \gamma + O\left(\frac{1}{x}\right).$$

Dirichlet Divisor Problem

Theorem

For $x > 0$,

$$D(x) := \sum_{n \le x}{}' d(n) = x(\log x + 2\gamma - 1) + \frac{1}{4} + \Delta(x), \qquad (7)$$

where γ *is Euler's constant, and* $\Delta(x)$ *is the "error term." Then, as* $x \to \infty$,

$$\Delta(x) = O(\sqrt{x}). \qquad (8)$$

The *Dirichlet divisor problem* asks for the correct order of magnitude of $\Delta(x)$ as $x \to \infty$.

The Bessel function of imaginary argument $Y_\nu(z)$ is defined by

$$Y_\nu(z) := \frac{J_\nu(z)\cos(\nu\pi) - J_{-\nu}(z)}{\sin(\nu\pi)}, \quad |z| < \infty, \tag{9}$$

and the modified Bessel function $K_\nu(z)$ is defined by

$$K_\nu(z) := \frac{\pi}{2}\,\frac{e^{\pi i\nu/2}J_{-\nu}(iz) - e^{-\pi i\nu/2}J_\nu(iz)}{\sin(\nu\pi)}, \quad -\pi < \arg z < \tfrac{1}{2}\pi. \tag{10}$$

If ν is an integer n, it is understood that we define the functions by taking the limits as $\nu \to n$ in (9) and (10).

Voronoï's Formula

$$\sum_{n\le x}{}' d(n) = x(\log x + 2\gamma - 1) + \frac{1}{4} + \sum_{n=1}^{\infty} d(n) \left(\frac{x}{n}\right)^{1/2} I_1(4\pi\sqrt{nx}), \tag{11}$$

where $x > 0$, γ denotes Euler's constant, and $I_1(z)$ is defined by

$$I_\nu(z) := -Y_\nu(z) - \frac{2}{\pi} K_\nu(z). \tag{12}$$

As $x \to \infty$,

$$Y_\nu(x) = \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{\pi\nu}{2} - \frac{\pi}{4}\right) + O\left(\frac{1}{x^{3/2}}\right), \tag{13}$$

$$K_\nu(x) = \sqrt{\frac{\pi}{2x}} e^{-x} + O\left(e^{-x}\frac{1}{x^{3/2}}\right). \tag{14}$$

Notes

Examining the asymptotic formulas (13) and (14), we conclude, as before, that the series on the right-hand side of (17) converges conditionally but not absolutely. Since the establishing of (17) in 1904, Voronoï's formula has been the starting point for most attempts at finding an upper bound for $\Delta(x)$.

Voronoï's Bound for $\Delta(x)$

In deriving the first improvement on Dirichlet's upper bound for $\Delta(x)$, in 1904, Voronoï proved that

$$\Delta(x) = O(x^{1/3} \log x), \qquad x \to \infty.$$

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Second Identity

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Ramanujan's Second Identity

Entry (p. 335)

Let $F(x)$ be defined by (6). Then, for $x > 0$ and $0 < \theta < 1$,

$$\sum_{n=1}^{\infty} F\left(\frac{x}{n}\right)\cos(2\pi n\theta) = \frac{1}{4} - x\log(2\sin(\pi\theta)) \tag{16}$$
$$+ \frac{1}{2}\sqrt{x}\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\left\{\frac{I_1\left(4\pi\sqrt{m(n+\theta)x}\right)}{\sqrt{m(n+\theta)}} + \frac{I_1\left(4\pi\sqrt{m(n+1-\theta)x}\right)}{\sqrt{m(n+1-\theta)}}\right\},$$

where

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$$D(x) = \sum_{d \le x}{}' \left[\frac{x}{d} \right],$$

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$$\sum_{n \le x}{}' d(n) = x\left(\log x + 2\gamma - 1\right) + \frac{1}{4} + \sum_{n=1}^{\infty} d(n) \left(\frac{x}{n} \right)^{1/2} I_1(4\pi\sqrt{nx}), \tag{17}$$

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- BCB, S. Kim and A. Zaharescu, *The circle and divisor problems, and double series of Bessel*

The Final Problem

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Sherlock Holmes' famous sidekick was Dr. Watson.

Final Problem for George Andrews and Myself

The identity involving the divisor function $d(n)$ on page 335 of Ramanujan's Lost Notebook.

This is OUR Final Problem.

The Final Problem
Has Been Solved

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B. C. Berndt, J. Li, and A. Zaharescu, *The final problem: an identity from Ramanujan's lost notebook*, J. London Math. Soc. **100** (2019), 568–591.

B. C. Berndt, J. Li, and A. Zaharescu, *The Final Problem: A Series Identity From The Lost Notebook*, to appear in: **George Andrews – 80 Years of Combinatory Analysis** (Trends in Mathematics), K. Alladi, B. C. Berndt, W. Y. C. Chen, P. Paule, J. Sellers, and A. J. Yee, eds., Springer Nature, to be published on 31-12-2020.

Junxian Li and Alexandru Zaharescu

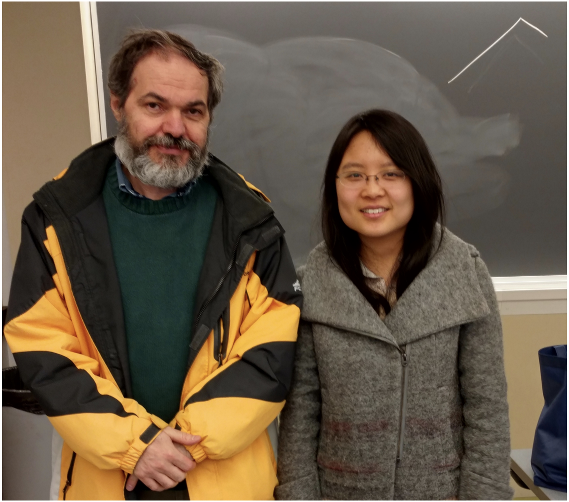

Figure: Alexandru Zaharescu and Junxian Li

Ideas from Previous Papers

Use ideas from

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4. We show that the series converges uniformly with respect to θ in any compact subinterval in $(0,1)$, provided that

$$\mathrm{Re}(s) > \tfrac{1}{4}, \mathrm{Re}(w) > \tfrac{1}{4}, \mathrm{Re}(s) + \mathrm{Re}(w) > \tfrac{25}{26}, \quad \text{if } x \text{ is an integer,}$$

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5. Must consider separately the case when x is an integer or not an integer.
6. Must divide the intervals $(0,\infty)$ for each summation variable in intervals for both "small" and "large" values of m and n.

Sketch of the Proof, Continued

1. There are discontinuities on both sides of (16). We want to eliminate them. We multiply both sides by $\sin^2(\pi\theta)$. Thus both sides are continuous on $[0, 1]$.

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2. Isolate the series of Bessel functions on the right-hand side.
3. We calculate the Fourier series on both sides of the amended identity. We show that the Fourier series are identical.
4. Because both sides of our identity are continuous, we can appeal to the uniqueness theorem for Fourier series to conclude that the functions are identical.

Are these Two Formulas of Ramanujan Isolated?

- $x^2 + y^2$ is a positive definite quadratic form. Might there exist similar formulas for other positive definite quadratic forms?

Dirichlet's Formula

Theorem

Let $\left(\frac{\cdot}{m}\right)$ denote the Kronecker symbol, where m is any positive integer. Let n be a positive integer coprime with the discriminant d of a positive-definite primitive integral binary quadratic form. Let $r(n)$ denote the number of all representations of n by a representative set of positive-definite primitive integral binary quadratic forms of discriminant d. Then

$$r(n) = w \sum_{k|n} \left(\frac{d}{k}\right), \tag{18}$$

where

$$w = \begin{cases} 2, & \text{if } \; d < -4, \\ 4, & \text{if } \; d = -4, \\ 6, & \text{if } \; d = -3. \end{cases} \tag{19}$$

In particular, for $d = -3$, there is only one class of positive-definite primitive integral binary quadratic forms of discriminant -3, and it is represented by $x^2 + xy + y^2$. We provide a formula for $r_3(n)$, the number of representations of n by the form $x^2 + xy + y^2$, analogous to (**??**).

An Analogue of Ramanujan's identity for $r_3(n)$

Theorem

If $x > 0$ and $r_3(n)$ counts the number of representations of n by the quadratic form $x^2 + xy + y^2$, where $3 \nmid n$. Then,

$$\sum_{n\le x}{}' r_3(n) = \frac{4\sqrt{3}\,\pi}{3}x - 3$$

$$+2\sqrt{3x}\left(\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\left\{\frac{J_1\left(4\pi\sqrt{m(n+\frac{1}{6})x}\right)}{\sqrt{m(n+\frac{1}{6})}} - \frac{J_1\left(4\pi\sqrt{m(n+1-\frac{1}{6})x}\right)}{\sqrt{m(n+1-\frac{1}{6})}}\right\}\right.$$

Trigonometric Sums

Let
$$\mathbb{SS}(\sigma,\theta,x) := \sum_{mn\le x}{}' mn \sin(2\pi m\sigma)\sin(2\pi n\theta). \tag{20}$$

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Assume that χ_1 and χ_2 are non-principal primitive odd characters modulo p and q, respectively. Let

$$\sigma = \frac{a}{p}, \quad (a,p)=1, \qquad \theta = \frac{b}{q}, \quad (b,q)=1.$$

Define

$$\mathbb{D}^*(p,q,x) := \sum_{n\le x}{}' nd_{\chi_1,\chi_2}(n). \tag{21}$$

where

$$d_{\chi_1,\chi_2}(n) := \sum_{d|n}\chi_1(d)\chi_2(n/d).$$

Trigonometric Sums, Cont.

$$\mathbb{SS}(\sigma,\theta,x) =$$

$$-\frac{1}{\phi(p)\phi(q)} \sum_{\substack{\chi_1 \bmod p \\ \chi_1 \text{ odd}}} \sum_{\substack{\chi_2 \bmod q \\ \chi_2 \text{ odd}}} \chi_1(\sigma)\chi_2(\theta)\tau(\overline{\chi_1})\tau(\overline{\chi_2})\mathbb{D}^*(p,q,x).$$

Note that $\mathbb{SS}(x)$ is a linear combination of the sums $\mathbb{D}^*(p,q,x)$.

Conjecture for $\mathbb{SS}(\sigma, \theta, x)$

We offer a conjectured Ω-theorem for $\mathbb{SS}(\sigma, \theta, x)$.

Conjecture

If $\mathbb{SS}(x)$ *is defined by* (20), *then*

$$\varlimsup_{x\to\infty} \frac{\mathbb{SS}(\sigma, \theta, x)}{x^{5/4}} = \infty,$$
$$\varliminf_{x\to\infty} \frac{\mathbb{SS}(\sigma, \theta, x)}{x^{5/4}} = -\infty.$$

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Conjecture

If $\mathbb{SS}(x)$ *is defined by* (20). *Then, for every* $\epsilon > 0$, *as* $x \to \infty$,

$$\mathbb{SS}(\sigma, \theta, x) = O(x^{5/4+\epsilon})$$

Theorem for $\mathbb{SS}(\sigma, \theta, x)$

Theorem

As $x \to \infty$, for every $\epsilon > 0$,

$$\mathbb{SS}(\sigma, \theta, x) = O(x^{17/12+\epsilon}).$$

$$\frac{17}{12} = 1.4166\ldots, \qquad \frac{5}{4} = 1.25.$$

Special Case: A Lattice Point Problem

Let $\theta = \sigma = \frac{1}{4}$. Then

$$\sin(2\pi n/4) = \begin{cases} (-1)^{(n-1)/2}, & n \text{ odd}, \\ 0, & n \text{ even}. \end{cases}$$

Then

$$\begin{aligned} \mathbb{SS}(\tfrac{1}{4}, \tfrac{1}{4}, x) &= -\sum_{\substack{mn \le x \\ m,n \text{ odd}}} mn(-1)^{(m+n)/2} \\ &= \sum_{(2j+1)(2k+1) \le x} (-1)^{j+k}(2j+1)(2k+1), \qquad (22) \end{aligned}$$

where we set $m = 2j+1$, $n = 2k+1$. This is a rather interesting lattice point problem. We are counting lattice points under the hyperbola $ab \le x$, but we require both coordinates to be odd and we put a weight on them.

Multivariable sin Sum

We can prove an identity for the sum

$$\begin{aligned}
&\mathbb{S}(a_1, a_2, \ldots a_k; p_1, p_2, \ldots p_k) \\
&:= \sideset{}{'}\sum_{0 \le n_1 n_2 \cdots n_k \le x} n_1 n_2 \cdots n_k \\
&\times \sin(2\pi n_1 a_1/p_1) \sin(2\pi n_2 a_2/p_2) \cdots \sin(2\pi n_k a_k/p_k).
\end{aligned}$$

References

BCB, S. Kim and A. Zaharescu, *Weighted divisor sums and Bessel function series, III*, J. Reine Angew. Math. **683** (2013), 67–96.

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Current research by BCB, Martino Fassina, Sun Kim, Alexandru Zaharescu.

General Setting: Chandrasekaran and Narasimhan

$$0 < \lambda_1 < \lambda_2 < \cdots \lambda_n \to \infty,$$
$$0 < \mu_1 < \mu_2 < \cdots \mu_n \to \infty$$

$$\{a_n\}, \quad \{b_n\}, \quad 1 \le n < \infty$$

$$\varphi(s) := \sum_{n=1}^{\infty} \frac{a_n}{\lambda_n^s}, \qquad \psi(s) = \sum_{n=1}^{\infty} \frac{b_n}{\mu_n^s}, \quad s = \sigma + it.$$

Assume that $\phi(s)$ and $\psi(s)$ satisfy a functional equation, for some $r > 0$,

$$\chi(s) := (2\pi)^{-s}\Gamma(s)\varphi(s) = (2\pi)^{s-r}\Gamma(r-s)\psi(r-s) \tag{23}$$

General Setting: Chandrasekaran and Narasimhan, continued

Theorem

Let $x > 0$. Let D be a domain which is the exterior of a bounded closed set S. In this domain, $\chi(s)$ is holomorphic (more hypotheses to be put on χ) . Let

$$P(x) := \frac{1}{2\pi i} \int_C \chi(s)(2\pi)^s x^{-s} ds, \tag{24}$$

where C is a curve or curves in D encircling all of S. Then, the functional equation (23) *implies the modular relation*

$$\sum_{n=1}^{\infty} a_n e^{-\lambda_n x} = P(x) + \left(\frac{2\pi}{x}\right)^r \sum_{n=1}^{\infty} b_n e^{-4\pi^2 \mu_n / x}. \tag{25}$$

Conversely, (25) *implies the functional equation* (23)*.*

General Setting: Chandrasekaran and Narasimhan, continued

Theorem

Let $x > 0$. *Let* $\rho > 2\sigma_a^* - r - \frac{1}{2}$, *where* σ_a^* *is the abscissa of absolute convergence of the Dirichlet series* $\psi(s)$. *Let*

$$Q_\rho(x) := \frac{1}{2\pi i} \int_C \frac{\chi(s)(2\pi)^s x^{s+\rho} ds}{\Gamma(\rho+1+s)}.$$

Then, the functional equation (23) *implies the identity*

$$\frac{1}{\Gamma(\rho+1)} \sum_{\lambda_n \le x}{}' a_n (x-\lambda_n)^\rho = Q_\rho(x)$$

$$+ \left(\frac{1}{2\pi}\right)^\rho \sum_{n=1}^\infty \left(\frac{x}{\mu_n}\right)^{(r+\rho)/2} b_n J_{r+\rho}\{4\pi\sqrt{\mu_n x}\} + Q_\rho(x), \qquad (26)$$

General Setting: Chandrasekaran and Narasimhan, continued

Theorem

(continued) where $J_\nu(x)$ denotes the ordinary Bessel function of order ν. Conversely, the Riesz sum identity (12) *implies the functional equation* (23)*.*

Theorem

The functional equation (23)*, the modular relation* (25)*, and the Riesz sum identity* (12) *implies the other two identities.*

In many cases, the Riesz sum identity is valid for $\rho > 2\sigma_a^* - r - \frac{3}{2}$.

Example: Ramanujan's Arithmetical Function $a_n = \tau(n)$

Let

$$f(s) = \sum_{n=1}^{\infty} \tau(n) n^{-s}, \quad \sigma > 13/2.$$

$$(2\pi)^{-s}\Gamma(s)f(s) = (2\pi)^{-(12-s)}\Gamma(12-s)f(12-s)$$

$$\sum_{n=1}^{\infty} \tau(n) e^{-2\pi n y} = \left(\frac{2\pi}{y}\right)^{12} \sum_{n=1}^{\infty} \tau(n) e^{-2\pi n/y}, \quad y > 0$$

For $\rho > -\frac{1}{2}$,

$$\frac{1}{\Gamma(\rho+1)} \sum_{n \le x}{}' \tau(n)(x-n)^{\rho}$$

$$= \left(\frac{1}{2\pi}\right)^{\rho} \sum_{n=1}^{\infty} \left(\frac{x}{n}\right)^{6+\rho/2} \tau(n) J_{12+\rho}\{4\pi\sqrt{nx}\}. \quad (27)$$

Convergence for Riesz Sum for $\tau(n)$

1. $\rho > 0$. The series on the right side of (27) is uniformly convergent in any interval $x_1 \leq x \leq x_2 < \infty$.
2. $\rho = 0$. The series is boundedly convergent in such an interval and uniformly convergent in such an interval that has no integral values.
3. $-\frac{1}{2} < \rho < 0$. The series is uniformly convergent in any interval that excludes integral values of x.

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K. Chandrasekharan and R. Narasimhan, *Hecke's functional equation and arithmetical identities*, Ann. Math. **74** (1961), 1–23.

Questions

- Do there exist analogues of Ramanujan's two series identities for other arithmetical functions, e.g., $\tau(n)$, that are generated by Dirichlet series satisfying a functional equation (23)?

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- How can we use Ramanujan's identities with the "extra" parameter to attack the famous circle and divisor problems?

Thank you very much for your kind invitation.