Ramanujan's lost notebook, q-series, and mock theta functions

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Overview of Lecture 1

- Ramanujan, First and Last Letters to Hardy, the Lost Notebook
- Partition function $p(n)$
  - asymptotic and exact formulas (Hardy-Ramanujan-Rademacher)
  - partition congruences (Ono)
- Freeman Dyson's rank of a partition
  - Rank Differences (Atkin and Swinnerton-Dyer)
  - Generalized rank differences (Bringmann-Ono)
  - The Crank (Andrews-Garvan)
- Modular forms, theta functions, and mock theta functions
  - Theta functions and modularity
  - Classical and modern definitions of mock theta functions
  - Radial limits
- Classical problems for mock theta functions
  - Asymptotic and exact formulas for Fourier coefficients (Dragonette; Andrews; Bringmann and Ono)
  - Proving identities between mock theta functions (Hickerson; Andrews, Garvan; Choi)
  - Ramanujan's vague definition and radial limits (Ono et al)
  - Modular transformation properties (Watson, Zwegers)
- Coming attractions
  - Appell–Lerch functions, Hecke-type double-sums, and a heuristic
  - Kronecker-type identities and sums of squares and triangular numbers
  - Threefield Identities, quantum modular forms
  - Modularity
  - Mock modularity

Highlights:

- Early life
- Student days
- Working as a clerk
- First letter to Hardy
- Life at Cambridge
- Last letter to Hardy
- The Lost Notebook

Book:
The Man Who Knew Infinity by Robert Kanigel
Movie:
The Man Who Knew Infinity

Ramanujan: Early Life

- Born 22 December 1887 in Erode, a town near Chennai, formerly known as Madras
- Grew up in Kumbakonam, a small town near Chennai, where his father was a clerk
- A stellar student, won many math awards at Town High School in Kumbakonam
- Won a scholarship to Government College
- Given a copy of G. S. Carr's *Synopsis of elementary results in pure mathematics*, which G. H. Hardy later described as

  "*...the 'synopsis' it professes to be. It contains enunciations of 6165 theorems, systematically and quite scientifically arranged, with proofs which are often little more than cross-references...*"
- Ramanujan became infatuated with mathematics, recording his "discoveries", imitating Carr's format. He typically offered no proofs of any kind, and his addiction made it impossible to focus on his school work at Government College
- Ramanujan unceremoniously flunked out

- Was given a second chance, winning a scholarship to attend Pachaiyappa's College in Madras
- His obsession with math again kept him from his school work
- Ramanujan flunked out a second time.
- For the next few years Ramanujan continued his research in near isolation
  "*...Ramanujan would sit working on the pial (porch) of his house..., legs pulled into his body, a large slate spread across his lap, madly scribbling,... When he figured something out, he sometimes seemed to talk to himself, smile, and shake his head with pleasure.*"
- Secured a job as a clerk at the Madras Port Trust
- Encouraged by colleagues, Ramanujan wrote to mathematicians in England such as M. J. M. Hill, H. F. Baker, E. W. Hobson, and G. H. Hardy.

C. P. Snow elegantly recount Hardy's reaction to the letter:

"One morning in 1913, he (Hardy) found, among the letters on his breakfast table, a large untidy envelope decorated with Indian stamps. When he opened it...he found line after line of symbols. He glanced at them without enthusiasm. He was by this time...a world famous mathematician, and ... he was accustomed to receiving manuscripts from strangers. ...The script appeared to consist of theorems, most of them wild or fantastic... There were no proofs of any kind... A fraud or genius? ...is a fraud of genius more probable than an unknown mathematician of genius? ...He decided that Ramanujan was in terms of ...genius, in the class of Gauss and Euler..."

Cover page followed by about 11 pages of mathematics with over 100 results.

Dear Sir, 16 January 1913

I beg to introduce myself to you as a clerk in the Accounts Department of the Port Trust Office at Madras on a salary of only £20 per annum. I am now about 23 years of age. I have had no university education but I have undergone the ordinary school course. After leaving school I have been employing the spare time at my disposal to work at Mathematics. I have not trodden through the conventional regular course which is followed in a university course, but I am striking out a new path for myself. I have made a special investigation of divergence series in general and the results are termed by the local mathematicians as "startling."

. . . [discusses analytic continuation of $\int_0^\infty x^{n-1}e^{-x}dx = \Gamma(n)$ to the left half-plane]. . .

Very recently I came across a tract published by you styled in *Orders of Infinity* in page 36 of which I find a statement that no definite expression has been as yet found for the number of prime numbers less than any given number. I have found an expression which very nearly approximate to the real result, the error being negligible. I would request you to go through the enclosed papers. Being poor, if you are convinced that there is anything of value I would like to have my theorems published. I have not given the actual investigations nor the expressions that I get but I have indicated the lines on which I proceed. Being inexperienced I would very highly value any advice you give me. Requesting to be excused for the trouble I give you.

I remain, Dear Sir, Yours truly,

S. Ramanujan.

J. E. Littlewood :

"...must be true because no one would have the imagination to invent them."

Ramanujan: First Letter to Hardy

Hardy:

Dear Sir, 8 February 1913

I was exceedingly interested by your letter and by the theorems which you state. You will however understand that, before I can judge properly of the value of what you have done, it is essential that I should see proofs of some of your assertions.

Your results seem to me to fall into roughly 3 classes:

- there are a number of results which are already known, or are easily deducible from known theorems;
- there are results which, so far as I know, are new and interesting, but rather from their curiosity and apparent difficulty than their importance;
- there are results which appear to be new and important, but in which almost everything depends on the precise rigour of the methods of proof which you have used.

As instances of these 3 classes I may mention...

...I hope very much that you will send me as quickly as possible at any rate a few of your proofs, and follow this more at your leisure by a more detailed account of your work on primes and divergence seems to me quite likely that you have done a good deal of work worth publication...

I am Yours very truly,
G. H. Hardy

- Wrote over thirty papers
- Was elected a Fellow of the Royal Society (F. R. S.)
- Named a Fellow of Trinity College
- Was hailed as a national hero in India
- Fell ill and returned to south India in 1919.
- Wrote his last letter to Hardy dated 12 January 1920:

  "*I am extremely sorry for not writing you a single letter up to now...I discovered interesting functions recently which I call "Mock" ϑ-functions. Unflike the "False" ϑ-functions (studied by Prof. Rogers in his interesting paper) they enter into mathematics as beautifully as the ordinary theta functions. I am sending you with this letter some examples.*"
- Died 26 April 1920 in Madras at age 32.

Selberg & Hecke

Hardy

"Opinions may differ about the importance of Ramanujan's work, the kind of standard by which it should be judged, and the influence which it is likely to have on mathematics of the future. . . . He would probably have been a greater mathematician if he could have been caught and tamed in his youth. On the other hand he would have been less of a Ramanujan, and more of a European professor, and the loss might have been more than the gain. . . "

- In his last letter to Hardy, Ramanujan gave a list of seventeen functions which he called "mock theta functions."

$$f(q) := \sum_{n\geq 0} \frac{q^{n^2}}{(-q;q)_n^2} = 1 + \frac{q}{(1+q)^2} + \frac{q^4}{(1+q)^2(1+q^2)^2} + \dots$$
$$= 1 + q - 2q^2 + 3q^3 - 3q^4 + 3q^5 + \dots$$

- Each mock theta function was defined by Ramanujan as a q-series convergent for $|q| < 1$. He stated that they have certain asymptotic properties as q approaches a root of unity radially, similar to the properties of ordinary theta functions, but that they are not theta functions.
- He stated several identities relating some of the mock theta functions to each other.

In Ramanujan's Lost Notebook, more identities such as the famous "mock theta conjectures" for the fifth order functions were found.

3D Python Plot!

How do we picture the asymptotic properties as q approaches a root of unity radially?

$$f(q) := \sum_{n\geq 0} \frac{q^{n^2}}{(-q;q)_n^2} = 1 + \frac{q}{(1+q)^2} + \frac{q^4}{(1+q)^2(1+q^2)^2} + \dots$$

The denominators tells suggest that singularities occur when q approaches an even root of unity, e.g. $q \to \zeta_{2k}$, $\zeta_{2k} := e^{2\pi i/2k}$.

How do we plot this? Let us write $q := we^{2\pi ix}$ where w goes from 0 to 1^- and x gives the root of unity.

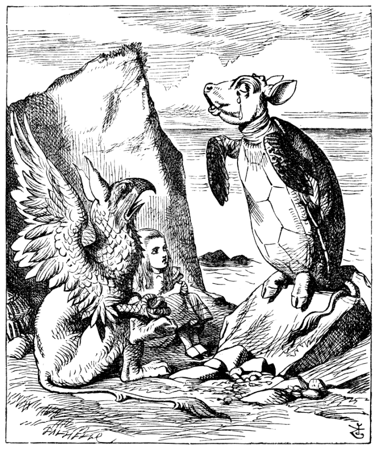
For example $x = 1/2$ corresponds to a 2nd root of unity, $x = 1/4$ or $3/4$ correspond to 4th roots of unity, $x = 1/6$ or $5/6$ correspond to a 6th roots of unity, etc.

We plot a wireframe in Python where the three-dimensional coordinates are given by

$$(w, x, |f(q)|).$$

Some roots of unity blow up faster than others.

Ramanujan: Last Letter to Hardy

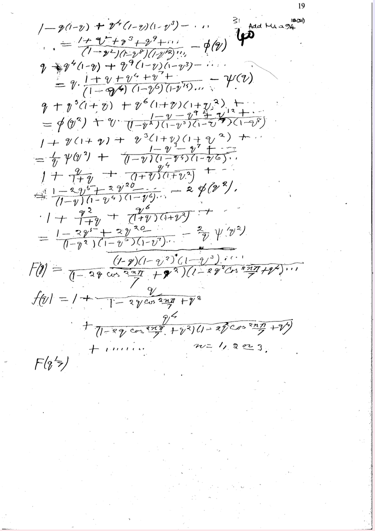


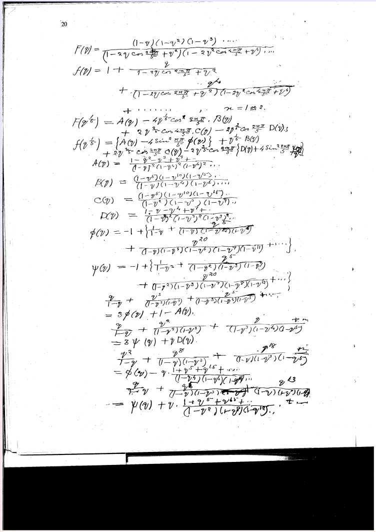
The Mock Turtle from Lewis Carroll's "Alice in Wonderland."

Ramanujan: The Lost Notebook

- For nearly 60 years, the only information on Ramanujan's mock theta functions was contained in the surviving portions of his last letter to Hardy.
- In 1976, George Andrews discovered the "lost notebook" in an old box of papers at Trinity College Library while on a visit to Cambridge University.
- The notebook contained over 100 pages of mathematical scrawl.
- Somehow survived a circuitous journey from India in 1920 to lie forgotten in the Trinity College Library Archives.
- In 1968, Rankin saved them from Watson's random papers just before they were about to be burned.
- Andrews:
  "*...the fact that its existence was never mentioned by anyone for over 55 years leads me to call it "lost"...*"

Ramanujan: The Lost Notebook





Topics List

- Ramanujan, First and Last Letters to Hardy, the Lost Notebook
- *we are here*
- Partition function $p(n)$
  - asymptotic and exact formulas (Hardy-Ramanujan-Rademacher)
  - partition congruences (Ono)
- Dyson's rank of a partition
- Modular forms, theta functions, and mock theta functions
- Classical problems for mock theta functions
- Coming Attractions

- A **partition** of a positive integer n is a weakly-decreasing sequence of positive integers whose sum is n.
- There are five partitions of the number 4: (4), $(3,1)$, $(2,2)$, $(2,1,1)$, $(1,1,1,1)$.

First few coefficients:

$$\begin{array}{ccccc}
p(0)=1, & p(1)=1, & p(2)=2, & p(3)=3, & p(4)=5, \\
p(5)=7, & p(6)=11, & p(7)=15, & p(8)=22, & p(9)=30, \\
p(10)=42, & p(11)=56, & p(12)=77, & p(13)=101, & p(14)=135, \\
p(15)=176, & p(16)=231, & p(17)=297, & p(18)=385, & p(19)=490, \\
p(20)=627, & p(21)=792, & p(22)=1002, & p(23)=1255, & p(24)=1575, \\
p(25)=1958, & p(26)=2436, & p(27)=3010, & p(28)=3718, & p(29)=4565, \\
p(30)=5604, & p(31)=6842, & p(32)=8349, & p(33)=10143, & p(34)=12310, \\
p(35)=14883, & p(36)=17977, & p(37)=21637, & p(38)=26015, & p(39)=31185, \\
p(40)=37338, & p(41)=44583, & p(42)=53174, & p(43)=63261, & p(44)=75175, \\
p(45)=89134, & p(46)=105558, & p(47)=124754, & p(48)=147273, & p(49)=173525,
\end{array}$$

p(50)=204226, ...
p(200)=3,972,999,029,388, ...
p(1000)= 24,061,467,864,032,622,473,692,149,727,991

depending on how columns are arranged, patterns emerge...

Ramanujan and partitions

- A **partition** of a positive integer n is a weakly-decreasing sequence of positive integers whose sum is n.
- There are five partitions of the number 4: (4), $(3,1)$, $(2,2)$, $(2,1,1)$, $(1,1,1,1)$.
- Ramanujan's congruences:

$$\begin{aligned} p(5n+4) &\equiv 0 \pmod{5}, \\ p(7n+5) &\equiv 0 \pmod{7}, \\ p(11n+6) &\equiv 0 \pmod{11}. \end{aligned}$$

The number of partitions of n is denoted by $p(n)$ with the generating function given by

$$\prod_{k=1}^{\infty} \frac{1}{(1-q^k)} = \sum_{n=0}^{\infty} p(n)q^n, \ \frac{1}{1-q^k} = 1 + q^k + q^{2k} + q^{3k} + \cdots$$

Two main questions:

- Asymptotic and exact formulas for $p(n)$
- Proofs and generalizations of Ramanujan's congruences

Ramanujan and partitions: Hardy-Ramanujan-Rademacher

We have the Hardy-Ramanujan asymptotic formula

$$p(n) \sim \frac{1}{4n\sqrt{3}} \cdot e^{\pi\sqrt{2n/3}}$$

p(50)∼217,590.1. . . (actual p(50)=204,226)
p(200)∼ 4,100,251,432,187.8. . . (actual p(200)=3,972,999,029,388)

On version of Rademacher's formula reads

$$p(n) = 2\pi(24n-1)^{-\frac{3}{4}} \sum_{k=1}^{\infty} \frac{A_k(n)}{k} \cdot I_{\frac{3}{2}}\left(\frac{\pi\sqrt{24n-1}}{6k}\right)$$

where $I_s(x)$ is the usual I-Bessel function of order s, $e(x) := e^{2\pi ix}$, and

$$A_k(n) := \frac{1}{2}\sqrt{\frac{k}{12}} \sum_{\substack{x \pmod{24k} \\ x^2 \equiv -24n+1 \pmod{24k}}} \left(\frac{12}{x}\right) \cdot e\left(\frac{x}{12k}\right)$$

Convergence is fast! The First eight terms yield

p(200)∼ 3,972,999,029,388.004 (actual p(200)=3,972,999,029,388)

Proofs use circle method and modularity.

Ramanujan and partitions: Ono

Question: Are there many congruences of the form $p(an+b) \equiv 0 \pmod{m}$?

Example: (Atkin and O'Brien)

$$p((11^3 \cdot 13)n + 237) \equiv 0 \pmod{13}$$

Conjecture: (Erdös) If m is prime, then there is at least one non-negative integer n_m for which

$$p(n_m) \equiv 0 \pmod{m}.$$

Theorem: (Ono) Let $m \geq 5$ be prime and let k be a positive integer. A positive proportion of primes ℓ have the property that

$$p\Big(\frac{m^k \ell^3 n + 1}{24}\Big) \equiv 0 \pmod{m}.$$

for every non-negative integer n coprime to ℓ.

Example: For $\ell = 59$, $m = 13$, $k = 1$, and considering integers $n \equiv 1 \pmod{24 \cdot 59}$, we have for every non-negative integer r, that

$$p((59^4 \cdot 13)r + 111247) \equiv 0 \pmod{13}.$$

Corollary: Conjecture of Erdös is true.

Ramanujan: "I have proved a number of arithmetical properties of $p(n)$... in particular that

$$p(5n+4) \equiv 0 \pmod 5$$

and

$$p(7n+5) \equiv 0 \pmod 7$$

I have since found another method which enables me to prove all of the properties and a variety of others, of which the most striking is

$$p(11n+6) \equiv 0 \pmod{11}$$

It appears that there are no equally simple properties for any moduli involving primes other that these three (i.e. $m = 5, 7, 11$)."

Theorem: (Ahlgren and Boylan) The only primes ℓ for which

$$p(\ell n + \beta_\ell) \equiv 0 \pmod{\ell}$$

for all non-negative integers n and some fixed $\beta_\ell \in \mathbb{Z}$ dependent on ℓ, are the primes $\ell \in \{5, 7, 11\}$.

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- Dyson's rank of a partition
    - Rank Differences (Atkin and Swinnerton-Dyer)
    - Generalized rank differences (Bringmann-Ono, Hickerson-Mortenson)
    - The Crank (Andrews-Garvan)
- Modular forms, theta functions, and mock theta functions
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Dyson's rank of a partition

What about...

Freeman Dyson defined the **rank of a partition** to be the largest part minus the number of parts. The ranks of the five partitions of 4 are

$$p(5n+4) \equiv 0 \pmod 5?$$

Partition	Rank
4	$4-1=3$
$3+1$	$3-2=1$
$2+2$	$2-2=0$
$2+1+1$	$2-3=-1$
$1+1+1+1$	$1-4=-3$

The rank gives an equinumerous distribution of the partitions of 4 into the five distinct residue classes mod 5: 3, 1, 0, 4, 2.

Fourier coefficients of mock theta functions can sometimes be written in terms of ranks!

What about...

The **rank of a partition** is defined to be the largest part minus the number of parts. The ranks of the seven partitions of 5 are

$$p(7n+5) \equiv 0 \pmod 7?$$

Partition	Rank
5	$5-1=4$
$4+1$	$4-2=2$
$3+2$	$3-2=1$
$3+1+1$	$3-3=0$
$2+2+1$	$2-3=-1$
$2+1+1+1$	$2-4=-2$
$1+1+1+1+1$	$1-5=-4$

The rank gives an equinumerous distribution of the partitions of 7 into the seven distinct residue classes mod 7: 4, 2, 1, 0, 6, 5, 3.

The **rank of a partition** is defined to be the largest part minus the number of parts.

What happens to the rank when we look at partitions of numbers of the form $11n+6$? Let us try the case $n=0$. We have

Partition	Rank
6	$6-1=5$
$5+1$	$5-2=3$
$4+2$	$4-2=2$
$4+1+1$	$4-3=1$
$3+3$	$3-2=1$
$3+2+1$	$3-3=0$
$3+1+1+1$	$3-4=-1$
$2+2+2$	$2-3=-1$
$2+2+1+1$	$2-4=-2$
$2+1+1+1+1$	$1-5=-4$

One sees right away that not all residues classes modulo 11 can be obtained by the rank Here we have the residues classes for 5,3,2,1,1,0,10,10,9,7.

Dyson's rank of a partition: alternate notation

Freeman Dyson defined the **rank of a partition** to be the largest part minus the number of parts. The ranks of the five partitions of 4 are

Partition	Rank
4	$4-1=3$
$3+1$	$3-2=1$
$2+2$	$2-2=0$
$2+1+1$	$2-3=-1$
$1+1+1+1$	$1-4=-3$

The rank gives an equinumerous distribution of the partitions of 4 into the five distinct residue classes mod 5: 3, 1, 0, 4, 2.
If we define

$$N(a, M; n) := \text{ number of partitions of } n \text{ with rank} \equiv a \pmod{M},$$

then

$$N(0,5;4) = N(1,5;4) = N(2,5;4) = N(3,5;4) = N(4,5;4) = 1.$$

Dyson conjectured that the rank explains $p(5n+4) \equiv 0 \pmod 5$ and $p(7n+5) \equiv 0 \pmod 7$ but not $p(11n+6) \equiv 0 \pmod{11}$.

Dyson's rank of a partition: Atkin and Swinnerton-Dyer

We have

$$N(a, M; n) := \text{ number of partitions of } n \text{ with rank} \equiv a \pmod{M}$$

Dyson conjectured (for $M = 5$):

$$\begin{aligned} N(0,5;5n+4) &= N(1,5;5n+4) = N(2,5;5n+4) = N(3,5;5n+4) \\ &= N(4,5,5n+4) = p(5n+4)/5, \dots \end{aligned}$$

Defining the rank difference:

$$R(a, b, M, c, m) := \sum_{n=0}^{\infty} \Big(N(a, M; mn + c) - N(b, M; mn + c) \Big) q^n,$$

Atkin and Swinnerton-Dyer proved (for $M = m = 5$):

$$\begin{aligned} R(i,j,5,4,5) &= 0, \text{ with } i,j \in \{0,1,2,3,4\} \\ R(0,2,5,0,5) &= \frac{\prod_{n=1}^{\infty}(1-q^{5n})}{\prod_{n=1}^{\infty}(1-q^{5n-1})(1-q^{5n-4})}, \dots \end{aligned}$$

Analogous conjectures results for $M = m = 7$

Generalized by Bringman and Ono using mock modularity and weak harmonic Maass forms.

Dyson's rank of a partition: the crank

Ramanujan's congruences:

$$
\begin{aligned}
p(5n+4) &\equiv 0 \pmod{5},\\
p(7n+5) &\equiv 0 \pmod{7},\\
p(11n+6) &\equiv 0 \pmod{11}.
\end{aligned}
$$

Dyson conjectured and Atkin and Swinnerton-Dyer proved that the rank explains the first two congruences.

"*Whether these guesses are warranted by the evidence, I leave to the reader to decide. Whatever the final verdict of posterity may be, I believe the 'crank' is unique among arithmetical functions in having been named before it was discovered. May it be preserved from the ignominious fate of the planet Vulcan!*" (Freeman Dyson, Eureka, 1944)

Dyson's rank of a partition: the crank

Definition: (Andrews, Garvan) For a partition π, let
- $\ell(\pi)$ denote the largest part of π,
- $\omega(\pi)$ denote the number of ones in π,
- $\mu(\pi)$ denote the number of parts of π larger than $\omega(\pi)$.

The crank $c(\pi)$ is given by

$$c(\pi) = \begin{cases} \ell(\pi) & \text{if } \omega(\pi) = 0, \\ \mu(\pi) - \omega(\pi) & \text{if } \omega(\pi) > 0. \end{cases}$$

We have

Partition	$(\ell(\pi),\ \omega(\pi),\ \mu(\pi))$	Crank
4	$(4,0,1)$	4
$3+1$	$(3,1,1)$	0
$2+2$	$(2,0,2)$	2
$2+1+1$	$(2,2,0)$	-2
$1+1+1+1$	$(1,4,0)$	-4

The crank gives an equinumerous distribution of the partitions of 4 into the five distinct residue classes mod 5: 4, 0, 2, 3, 1. Explains all three of Ramanujan's congruences!

Definition: (Andrews, Garvan) For a partition π, let $\ell(\pi)$ denote the largest part of π, $\omega(\pi)$ denote the number of ones in π, $\mu(\pi)$ denote the number of parts of π larger than $\omega(\pi)$. The crank $c(\pi)$ is given by

$$c(\pi) = \begin{cases} \ell(\pi) & \text{if } \omega(\pi) = 0, \\ \mu(\pi) - \omega(\pi) & \text{if } \omega(\pi) > 0. \end{cases}$$

Partition	$(\ell(\pi),\ \omega(\pi),\ \mu(\pi))$	Crank
6	$(6,0,1)$	6
$5+1$	$(5,1,1)$	0
$4+2$	$(4,0,4)$	4
$4+1+1$	$(4,2,1)$	-1
$3+3$	$(3,0,3)$	3
$3+2+1$	$(3,1,2)$	1
$3+1+1+1$	$(3,3,0)$	-3
$2+2+2$	$(2,0,2)$	2
$2+2+1+1$	$(2,2,0)$	-2
$2+1+1+1+1$	$(2,4,0)$	-4
$1+1+1+1+1+1$	$(1,6,0)$	-6

Residue classes mod 11: 6,0,4,10,3,1,8,2,9,7,5. Explains all three of Ramanujan's congruences!

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Modular forms, theta functions, and mock theta functions

A modular form is a holomorphic function f in the complex upper half-plane $\mathfrak{h} := \{z \in \mathbb{C} \Big| Im(z) > 0\}$ satisfying the transformation equation

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$$

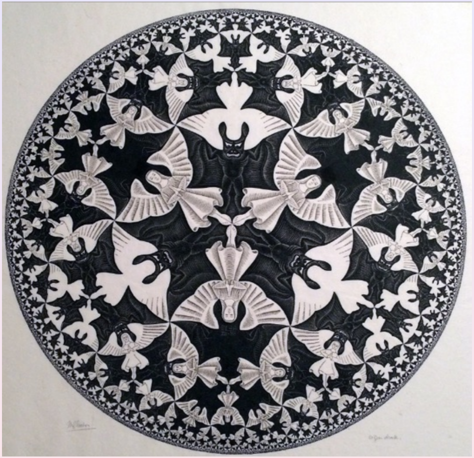
for all $z \in \mathfrak{h}$ and all matrices $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, where $k \in \mathbb{Z}$ is the weight of the modular form. Actually only need to check on generators $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$.

Modular forms satisfying a given set of transformation equations live inside the same finite-dimensional vector space. We then say $f(\tau) \in \mathrm{M}_k(\mathrm{SL}_2(\mathbb{Z}))$.

There are many variants

- meromorphic and weakly modular
- one can replace the group $SL_2(\mathbb{Z})$ with certain subgroups,
- the weight k may be half-integral or even rational,
- the function f can be vector-valued rather than scalar-valued,
- the function f may need a correction term to become modular,
- and more!

Modular forms, theta functions, and mock theta functions: symmetries



M.C. Escher's "Angels and Devils."

Rephrasing let us set $q := e^{2\pi i\tau}$ where $\tau \in \mathfrak{h}$, e.g. $|q| < 1$, which takes the upper half-plane to the punctured unit disc:

$$\mathfrak{h} := \{z \in \mathbb{Z} \Big| Im(z) > 0\} \to D^{\star} := \{q \in \mathbb{C} \Big| 0 < |q| < 1\}.$$

Modular forms have an extra condition of holomorphic at infinity.

The modular form $f(\tau)$ then has a Fourier expansion

$$f(\tau) = \sum_{n=0}^{\infty} a(n)q^n = a(0) + a(1)q^1 + a(2)q^2 + a(3)q^3 + \cdots$$

The Fourier coefficients $a(n)$ often store arithmetic information.

Finite-dimensional nature of $\mathrm{M}_k(\mathrm{SL}_2(\mathbb{Z}))$ allows one to prove identities for the $a(n)$'s.

Nontrivial example:

Eisenstein series!

For example, for k even, $k \geq 4$:

$$G_k(\tau) := \sum_{\substack{(c,d)\in\mathbb{Z}^2\\(c,d)\neq(0,0)}} \frac{1}{(c\tau+d)^k} \in \mathrm{M}_k(\mathrm{SL}_2(\mathbb{Z})).$$

In terms of a Fourier expansion, where $q := e^{2\pi i\tau}$ and $\tau \in \{x+iy \in \mathbb{C} \big| y>0\}$, e.g. $|q|<1$:

$$G_k(\tau) = 2\zeta(k)E_k(\tau), \text{ where } E_k(\tau) = 1 - \frac{2k}{B_k}\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n,$$

with $\zeta(k) = \sum_{n=1}^{\infty}\frac{1}{n^k}$, $\sigma_{k-1}(n) = \sum_{0<m|n} m^{k-1}$, and the B_k are the Bernoulli numbers, where

$$\frac{t}{e^t-1} = \sum_{k=0}^{\infty} B_k\frac{t^k}{k!}.$$

What about $k=2$?

Theta functions, modular forms, and mock theta functions

There are various ways to produce new modular forms, for example:
If $f_1(\tau) \in \mathrm{M}_{k_1}(\mathrm{SL}_2(\mathbb{Z}))$ and $f_2(\tau) \in \mathrm{M}_{k_2}(\mathrm{SL}_2(\mathbb{Z}))$, then $(f_1 f_2)(\tau) \in \mathrm{M}_{k_1+k_2}(\mathrm{SL}_2(\mathbb{Z}))$.

It turns out that $\dim_{\mathbb{C}} \mathrm{M}_8(\mathrm{SL}_2(\mathbb{Z})) = 1$, so

$$E_4(\tau)^2 = E_8(\tau).$$

Because (note $q := e^{2\pi i \tau}$, $|q| < 1$)

$$E_4(\tau) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n,$$

$$E_8(\tau) = 1 + 480 \sum_{n=1}^{\infty} \sigma_7(n) q^n,$$

we have

$$\sigma_7(n) = \sigma_3(n) + 120 \sum_{i=1}^{n-1} \sigma_3(i)\sigma_3(n-i), \ \ n \geq 1.$$

where

$$\sigma_3(n) = \sum_{0<m|n} m^3, \ \ \sigma_7(n) = \sum_{0<m|n} m^7.$$

Recall variants of modular forms:
Theta functions are examples of modular forms when the weight $k = 1/2$.

The theta function is defined (now we have z and τ)

$$\vartheta(z,\tau) := \sum_{n\in\mathbb{Z}} exp(\pi i n^2\tau + 2\pi i n z), \quad z \in \mathbb{C},\ \tau \in \mathbb{H}.$$

We can think of this as a Fourier series for a function in z, periodic with respect to $z \to z+1$.

With respect to $\mathbb{Z}\cdot 1 \bigoplus \mathbb{Z}\cdot\tau$, the theta function has elliptic transformation equations:

- $\vartheta(z+1,\tau) = \vartheta(z,\tau)$,
- $\vartheta(z+\tau,\tau) = exp(-\pi i\tau - 2\pi i z)\cdot\vartheta(z,\tau)$,
- Modular transformation properties (next slide).

(Alternate form) We write $q := exp(2\pi i\tau)$ and $x := exp(2\pi i z)$ to have

$$j(x;q) := \sum_{n=-\infty}^{\infty} (-1)^n q^{\binom{n}{2}} x^n = \prod_{n=1}^{\infty}(1-q^{n-1}x)(1-q^n/x)(1-q^n),$$

where the last equality follows from Jacobi's triple product identity.

Recall variants of modular forms:
Jacobi theta functions are examples of vector-valued transformation properties.
Here $w := e^{\pi i z}$ and $q := e^{\pi i \tau}$.

$$\vartheta_{00}(w,q) := \sum_{n=-\infty}^{\infty} (w^2)^n q^{n^2}, \qquad \vartheta_{01}(w,q) := \sum_{n=-\infty}^{\infty} (-1)^n (w^2)^n q^{n^2},$$

$$\vartheta_{10}(w,q) := \sum_{n=-\infty}^{\infty} (w^2)^{n+\frac{1}{2}} q^{(n+\frac{1}{2})^2}, \qquad \vartheta_{11}(w,q) := i \sum_{n=-\infty}^{\infty} (-1)^n (w^2)^{n+\frac{1}{2}} q^{(n+\frac{1}{2})^2}$$

Here we have

$$\begin{pmatrix} \vartheta_{00} \\ \vartheta_{01} \\ \vartheta_{10} \\ \vartheta_{11} \end{pmatrix} \left(\frac{z}{\tau}; -\frac{1}{\tau}\right) = \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & i\alpha \end{pmatrix} \begin{pmatrix} \vartheta_{00} \\ \vartheta_{01} \\ \vartheta_{10} \\ \vartheta_{11} \end{pmatrix} \left(z; \tau\right)$$

where

$$\alpha = (-i\tau)^{\frac{1}{2}} \exp\left(\frac{\pi}{\tau} i z^2\right).$$

How do we picture the asymptotic properties of Jacobi theta functions?

3D Python Plot!

Here $w := e^{\pi i z}$ and $q := e^{\pi i \tau}$:

$$\vartheta_1(z;q) := -\vartheta_{11}(w,q) = -i \sum_{n=-\infty}^{\infty} (-1)^n (w^2)^{n+\frac{1}{2}} q^{(n+\frac{1}{2})^2}$$
$$= 2q^{1/4} \sum_{n=0}^{\infty} (-1)^n q^{n^2+n} sin((2n+1)z)$$

- We let q go from $0 \to 1$ along the real axis.
- We let z range from $0 \to 2$.

We plot a wireframe in Python where the three-dimensional coordinates are given by

$$(z, q, f(q)).$$

Modular forms, theta functions, and mock theta functions

Recall variants of modular forms:
Dedekind's eta function is an example where the weight $k = 1/2$:
Here $q := e^{2\pi i\tau}$,

$$\eta(\tau) := e^{\frac{\pi i}{12}} \prod_{n=1}^{\infty}(1 - e^{2n\pi i\tau}) = q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1 - q^n).$$

Here the modular transformation properties read

$$\eta(\tau + 1) = e^{\frac{\pi i}{12}}\eta(\tau),$$
$$\eta(-\frac{1}{\tau}) = \sqrt{-i\tau}\eta(\tau).$$

Note that we only have to check the transformation properties on the generators of $SL_2(\mathbb{Z})$

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z),$$

where the generators are $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$.

Recall the generating function for the partition function:

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{n=1}^{\infty} \frac{1}{1-q^n}.$$

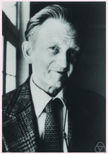
What does one mean by "like a theta function but not a theta function?"

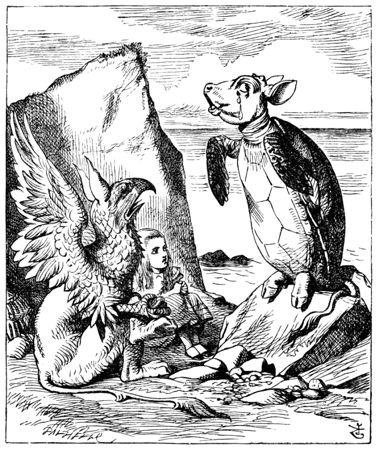
A mock theta function is a function $f(q)$ defined by a q-series which converges for $|q| < 1$ which satisfies

(0) For every root of unity ζ, there is a theta function $\theta_\zeta(q)$ such that the difference $f(q) - \theta_\zeta(q)$ is bounded as $q \to \zeta$ radially.

(1) There is no single theta function which works for all ζ; i.e., for every theta function $\theta(q)$ there is some root of unity ζ for which $f(q) - \theta(q)$ is unbounded as $q \to \zeta$ radially.

Condition (0) was proved for third and fifth order functions by Watson, and for seventh order functions by Selberg.





The Mock Turtle from Lewis Carroll's "Alice in Wonderland."

What is an example of one of the theta functions paired with the mock theta function $f(q)$?

3D Python Plot!

How do we picture the asymptotic properties as q approaches a root of unity radially?

$$f(q) := \sum_{n\geq 0} \frac{q^{n^2}}{(-q;q)_n^2} = 1 + \frac{q}{(1+q)^2} + \frac{q^4}{(1+q)^2(1+q^2)^2} + \dots$$

The theta functions paired with $f(q)$ are $\pm b(q)$ where

$$b(q) := \prod_{n=1}^{\infty} (1-q^{2n-1})^2(1-q^n).$$

Let us write $q := we^{2\pi ix}$ where w goes from 0 to 1^- and x gives the root of unity.

We plot a wireframe in Python where the three-dimensional coordinates are given by

$$(w, x, |b(q)|).$$

Some roots of unity blow up faster than others.

Ramanujan also made slightly more specific claims about asymptotic behaviour. We give an example. We recall the third order mock theta function

$$f(q) := \sum_{n\geq 0} \frac{q^{n^2}}{(-q;q)_n^2} = 1 + \frac{q}{(1+q)^2} + \frac{q^4}{(1+q)^2(1+q^2)^2} + \dots$$
$$= 1 + q - 2q^2 + 3q^3 - 3q^4 + 3q^5 + \dots$$

and the theta function

$$b(q) := \prod_{n=1}^{\infty}(1-q^{2n-1})^2(1-q^n).$$

Ramanujan: "*As q approaches an even root of unity of order $2k$, the difference $f(q)-(-1)^k b(q)$ is absolutely bounded.*"

Theorem: Folsom, Ono, Rhoades. If ζ is a primitive even order $2k$ root of unity, then, as q approaches ζ radially within the unit disk, we have that

$$\lim_{q\to\zeta}\Big(f(q)-(-1)^k b(q)\Big) = -4\cdot\sum_{n=0}^{k-1}(1+\zeta)^2(1+\zeta^2)^2\cdots(1+\zeta^n)^2\zeta^{n+1}.$$

Here the right-hand side is

$$U(\omega;q) := \sum_{n=0}^{\infty}(\omega q;q)_n(q\omega;q)_n q^{n+1}.$$

Modular forms, theta functions, and mock theta functions: definitions

Berndt: "*It has not been proved that any of Ramanujan's mock theta functions are really mock theta functions according to his definition.*"

Theorem: (Griffin, Ono, Rolen) Ramanujan's mock theta functions satisfy his definition!

Theorem: (Rhoades) Ramanujan's definition of a mock theta function is not equivalent to the modern definition of a mock theta function.

Two functions

$$V_1(q) := \frac{1}{(q)_\infty}\left(\frac{1}{12} - \sum_{n\neq 0}\frac{q^n}{(1-q^n)^2}\left(3+(-1)^n q^{\frac{3n^2-n}{2}}(1+q^n)\right)\right)$$

$$V_2(q) := \frac{1}{(q)_\infty}\left(\frac{1}{12} - 2\sum_{n\neq 0}\frac{nq^n}{(1-q^n)^2}\left(1+(-1)^{n+1} q^{\frac{1}{2}n(n-1)}\right)\right)$$

Either $V_1(q)$ is a mock theta function according to the modern definition, but not Ramanujan's, or $V_2(q)$ is a mock theta function according to Ramanujan's definition, but not the modern definition.

Modular forms, theta functions, and mock theta functions: timeline

- Ramanujan's last letter 1920
- Watson 1937
- Selberg 1938
- Dragonette PhD Thesis 1952
- Andrews PhD Thesis 1966
- Ramanujan's Lost Notebook 1976
- Hickerson's proof of mock theta conjectures 1988
- Hickerson's proof of 7th order identities 1988
- Choi's proof of 10th order identities 1999
- Zwegers' PhD Thesis 2002
- Bruinier and Funke 2004
- Bringmann and Ono proof of Dragonette conjecture 2006
- Bringmann and Ono work on Dyson's ranks 2010
- Griffin, Ono, and Rolen and Ramanujan's definition 2013

Topics List

- Ramanujan, First and Last Letters to Hardy, the Lost Notebook
- Partition function $p(n)$
- Dyson's rank of a partition
- Modular forms, theta functions, and mock theta functions
- *we are here*
- Classical problems for mock theta functions
  - Asymptotic and exact formulas for Fourier coefficients (Dragonette; Andrews; Bringmann and Ono)
  - Proving identities between mock theta functions (Watson; Hickerson; Andrews, Garvan; Choi)
  - Ramanujan's vague definition and radial limits (Watson; Selberg; Ono et al; Zudilin; Mortenson)
  - Modular transformation properties (Watson; Zwegers)
- Coming Attractions

Notation:

Let q be a complex number with $0 < |q| < 1$.

Finite and infinite products:

$$(x)_n = (x;q)_n := \prod_{i=0}^{n-1}(1-q^ix), \quad (x)_\infty = (x;q)_\infty := \prod_{k=0}^{\infty}(1-q^kx).$$

The theta function

$$j(x,q) := \sum_{n=-\infty}^{\infty}(-1)^nq^{\binom{n}{2}}x^n = (x)_\infty(q/x)_\infty(q)_\infty,$$

where

$$j(x_q, x_2, ..., x_n; q) := j(x_1;q)j(x_2;q)\cdots j(x_n;q)$$

and

$$J_{a,m} := j(q^a;q^m),\ \overline{J}_{a,m} := j(-q^a;q^m),\ J_m := J_{m,3m} = \prod_{k=1}^{\infty}(1-q^{nk}).$$

Classical problems, forms, and techniques: notation

Let q be a complex number with $0 < |q| < 1$.

Finite and infinite products:

$$(x)_n = (x;q)_n := \prod_{i=0}^{n-1}(1-q^i x), \quad (x)_\infty = (x;q)_\infty := \prod_{k=0}^{\infty}(1-q^k x).$$

A series $\sum x_n$ is called a q-hypergeometric series if the ratio of successive terms x_{n+1}/x_n is a rational function function of q.

Example:

$$f(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q;q)_n^2} = 1 + \frac{q}{(1+q)^2} + + \frac{q^4}{(1+q)^2(1+q^2)^2} + \dots,$$

where the ratio of successive terms is

$$\frac{x_{n+1}}{x_n} = \frac{q^{(n+1)^2}}{(-q;q)_{n+1}^2} \cdot \frac{(-q;q)_{n+1}^2}{q^{n^2}} = q^{2n+1}(1+q^{n+1})^2.$$

Classical problems, forms, and techniques: notation

The Lost Notebook–Rosetta Stone for q-hypergeometric series

Scores of entries in which q-hypergeometric series are expressed as

- Appell-Lerch functions (mock theta functions)

$$f(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q;q)_n^2} = 4m(-q,q^3,q) + \frac{(q^3;q^6)_\infty^2 (q^3;q^3)_\infty^2}{(q;q)_\infty},$$

  where the Appell-Lerch function $m(x,q,z)$ is defined

$$m(x,q,z) := \frac{1}{j(z;q)} \sum_{r=-\infty}^{\infty} \frac{(-1)^r q^{\binom{r}{2}} z^r}{1-q^{r-1}xz},\ j(z;q) = \sum_{n=-\infty}^{\infty} (-1)^n z^n q^{\binom{n}{2}},$$

- theta functions (Rogers-Ramanujan type)

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \frac{(q^2,q^3;q^5)_\infty}{(q;q)_\infty},$$

- partial theta functions (very mysterious)

$$\sum_{n=0}^{\infty} \frac{(q;q^2)_n^2 q^n}{(-q;q)_{2n+1}} = \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)}.$$

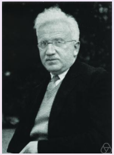
How do we relate the various generating functions to each other?

Classical problems, forms, and techniques

Classical problems for mock theta functions:

- Asymptotic and exact formulas for Fourier coefficients
- Identities between the mock theta functions (Hickerson; Andrews, Garvan; Choi)
- Radial Limits (Folsom, Ono, Rhoades; Zudilin; Mortenson)
- Modular transformation properties (Watson; Zwegers)

A common theme to solving problems is to change from one form to another. For example, in attempts to prove the mock theta conjectures and find modularity properties, Andrews introduced techniques to translate mock theta functions into a new representations: Hecke-type double sums and Fourier coefficients of meromorphic Jacobi forms.



$$f_0(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q)_n} = 1 + \frac{q}{1+q} + \frac{q^4}{(1+q)(1+q^2)} + \cdots$$

$$f_0(q) \cdot \prod_{n=1}^{\infty}(1-q^n) = \sum_{\substack{n\geq 0 \\ |j|\leq n}} (-1)^j q^{\frac{5}{2}n^2+\frac{1}{2}n-j^2}(1-q^{4n+2})$$

The Andrews-Dragonette Conjecture.

Recall the mock theta function

$$f(q) := \sum_{n\geq 0} \frac{q^{n^2}}{(-q;q)_n^2} = 1 + \frac{q}{(1+q)^2} + \frac{q^4}{(1+q)^2(1+q^2)^2} + \dots$$
$$= \sum_{n=0}^{\infty} \alpha(n) q^n = 1 + q - 2q^2 + 3q^3 - 3q^4 + 3q^5 - 5q^6 + \dots$$

Ramanujan (Claim), Dragonette (Proof):

$$\alpha(n) = (-1)^{n-1} \frac{exp(\pi\sqrt{\frac{n}{6} - \frac{1}{144}})}{2\sqrt{n - \frac{1}{24}}} + \mathcal{O}\Bigg(\frac{\exp\left(\frac{1}{2}\pi\sqrt{\frac{n}{6} - \frac{1}{144}}\right)}{\sqrt{n - \frac{1}{24}}}\Bigg)$$

Conjecture (Andrews-Dragonette), Theorem (Bringmann-Ono):

$$\alpha(n) = \pi(24n-1)^{-\frac{1}{4}} \sum_{k=1}^{\infty} \frac{(-1)^{\lfloor \frac{k+1}{2} \rfloor} A_{2k}(n - \frac{k(1+(-1)^k)}{4})}{k} \cdot I_{\frac{1}{2}}\left(\frac{\pi\sqrt{24n-1}}{12k}\right)$$

Proof uses mock modularity and weak harmonic Maass forms.

Examples of various forms for the fifth order function $f_0(q)$:

$$f_0(q)=\sum_{n\geq 0}\frac{q^{n^2}}{(-q)_n}=1+\frac{q}{1+q}+\frac{q^4}{(1+q)(1+q^2)}+\cdots$$

$$f_0(q)\cdot\prod_{n=1}^{\infty}(1-q^n)=\sum_{\substack{n\geq 0\\ |j|\leq n}}(-1)^j q^{\frac{5}{2}n^2+\frac{1}{2}n-j^2}(1-q^{4n+2})$$

$$\begin{aligned}f_0(q)&=m(q^{14},q^{30},q^{14})+m(q^{14},q^{30},q^{29})+q^{-2}m(q^4,q^{30},q^4)+q^{-2}m(q^4,q^{30},q^{19}).\\ &=2m(q^{14},q^{30},q^4)+2m(q^4,q^{30},q^4)+\frac{J_{5,10}J_{2,5}}{J_1}.\end{aligned}$$

The Appell-Lerch function is defined

$$m(x,q,z):=\frac{1}{j(z;q)}\sum_{r=-\infty}^{\infty}\frac{(-1)^r q^{\binom{r}{2}}z^r}{1-q^{r-1}xz},\quad j(z;q):=\sum_{n=-\infty}^{\infty}(-1)^n z^n q^{\binom{n}{2}}.$$

Classical problems, forms, and techniques: identities

For the third order functions

$$f(q) := \sum_{n\geq 0} \frac{q^{n^2}}{(-q)_n^2} \text{ and } \psi(q) := \sum_{n\geq 1} \frac{q^{n^2}}{(q;q^2)_n},$$

Ramanujan stated the following identity:

$$f(q) + 4\psi(-q) = \frac{J_1^3}{J_2^2}.$$

Using

$$1 + \sum_{n=1}^{\infty} \frac{(a, q\sqrt{a}, -q\sqrt{a}, b, c, d, e, f; q)_n}{(q, \sqrt{a}, -\sqrt{a}, aq/c, aq/d, aq/e, aq/f, aq/g; q)_n} \cdot \left(\frac{a^2q^2}{cdefg}\right)^n$$
$$= \frac{(aq, aq/fg, aq/ge, aq/ef; q)_\infty}{(aq/e, aq/f, aq/g, aq/efg; q)_\infty} \cdot \left[1 + \sum_{n=1}^{\infty} \frac{(aq/cd, e, f, g; q)_n}{(q, efg/a, aq/c, aq/d; q)_n} \cdot q^n\right],$$

Watson showed (slightly rewritten)

$$f(q) = 4m(-q, q^3, q) + \frac{J_{3,6}^2}{J_1} \text{ and } \psi(q) = -m(q, -q^3, -q) + \frac{qJ_{12}^3}{J_4 J_{3,12}}.$$

and as a result

$$f(q) + 4\psi(-q) = 4m(-q, q^3, q) - 4m(-q, q^3, q) + \frac{J_{3,6}^2}{J_1} - 4\frac{qJ_{12}^3}{J_4\overline{J}_{3,12}} = \frac{J_1^3}{J_2^2}.$$

Mock theta function identities (Andrews, Garvan; Hickerson). Noting

$$f_0(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q;q)_n}, \ f_1(q) := \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(-q;q)_n},$$

and

$$\Phi(q) := -1 + \sum_{n=0}^{\infty} \frac{q^{5n^2}}{(q;q^5)_{n+1}(q^4;q^5)_n}, \ \Psi(q) := -1 + \sum_{n=0}^{\infty} \frac{q^{5n^2}}{(q^2;q^5)_{n+1}(q^3;q^5)_n},$$

the (ex-) mock theta conjectures read

$$f_0(q) = \frac{(q^5;q^5)_\infty (q^5;q^{10})_\infty}{(q;q^5)_\infty (q^4;q^5)_\infty} - 2\Phi(q^2),$$
$$f_1(q) = \frac{(q^5;q^5)_\infty (q^5;q^{10})_\infty}{(q^2;q^5)_\infty (q^3;q^5)_\infty} - \frac{2}{q}\Psi(q^2).$$

Hickerson's proof (sketch). We want to prove

$$f_0(q)=\frac{(q^5;q^5)_\infty(q^5;q^{10})_\infty}{(q;q^5)_\infty(q^4;q^5)_\infty}-2\Phi(q^2)$$

If $|q|<1$ and z is not an integral power of q^2, let

$$B(z)=B(z;q):=\frac{z^2J_2j(-z;q)j(z;q^3)}{j(z;q^2)}.$$

Expand $B(z;q)$ two ways. For the first way we find that the coefficient of z^1 is

$$q\sum_{\substack{n\geq 0\\ |j|\leq n}}(-1)^jq^{\frac{5}{2}n^2+\frac{1}{2}n-j^2}(1-q^{4n+2})=q\cdot\prod_{n=1}^{\infty}(1-q^n)\cdot f_0(q).$$

For the second way we find that the coefficient of z^1 is

$$q\cdot\prod_{n=1}^{\infty}(1-q^n)\cdot\Big(\frac{(q^5;q^5)_\infty(q^5;q^{10})_\infty}{(q;q^5)_\infty(q^4;q^5)_\infty}-2\Phi(q^2)\Big).$$

The conjecture for $f_1(q)$ follows from the coefficients of z^2.

Define $F=(f_0,f_1,f_2)^T$ by $f_0(\tau)=q^{-\frac{1}{24}}f(q)$, $f_1(\tau)=2q^{\frac{1}{3}}\omega(q^{\frac{1}{2}})$, and $f_2(\tau)=2q^{\frac{1}{3}}\omega(-q^{\frac{1}{2}})$.

Watson showed

$$F(\tau+1)=\begin{pmatrix}\zeta_{24}^{-1} & 0 & 0\\ 0 & 0 & \zeta_3\\ 0 & \zeta_3 & 0\end{pmatrix}F(\tau)$$

and

$$\frac{1}{\sqrt{-i\tau}}F(-1/\tau)=\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & -1\end{pmatrix}F(\tau)+R(\tau),$$

with $\zeta_n=e^{2\pi i/n}$ and $R(\tau)=4\sqrt{3}\sqrt{-i\tau}(j_2(\tau),-j_1(\tau),j_3(\tau))^T$, where

$$j_1(\tau)=\int_0^\infty e^{3\pi i\tau x^2}\frac{\sinh 2\pi\tau x}{\sinh 3\pi\tau x}dx,$$

and $j_2(\tau)$ and $j_3(\tau)$ are defined similarly.

Zwegers's pathbreaking result!

Recall that $F(\tau) = (f_0, f_1, f_2)^T$, where the f_i's are in terms of $f(q)$ and $\omega(q)$.

Define

$$G(\tau) := 2i\sqrt{3}\int_{-\overline{\tau}}^{i\infty} \frac{(g_0(z), g_1(z), g_2(z))^T}{\sqrt{-i(z+\tau)}}\,dz$$

where

$$g_0(z) := \sum_{n\in\mathbb{Z}}(-1)^n(n+1/3)e^{3\pi i(n+\frac{1}{3})^2 z}$$

and $g_1(z)$ and $g_2(z)$ are defined similarly. Letting $H(\tau) := F(\tau) - G(\tau)$, Zwegers showed

$$H(\tau+1) = \begin{pmatrix} \zeta_{24}^{-1} & 0 & 0 \\ 0 & 0 & \zeta_3 \\ 0 & \zeta_3 & 0 \end{pmatrix} H(\tau)$$

and

$$\frac{1}{\sqrt{-i\tau}}H(-1/\tau) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} H(\tau).$$

Bruinier and Funke: weak harmonic Maass forms.

Zagier: "*with the knowledge of the transformation properties of the mock theta functions the proof becomes automatic: one only has to verify that the left- and right-hand sides of the identities become modular after the addition of the same non-holomorphic correction term and that the first few coefficients of the q-expansions agree. Moreover, knowing the transformation behavior also allows one to find new identities in a systematic way.* "

How many coefficients?
Sturm:

$$\frac{k}{12}[SL_2(\mathbb{Z}):\Gamma]$$

Check: Eisenstein series example. Recall $dim_{\mathbb{C}} M_8(SL_2(\mathbb{Z}))=1$, $E_4^2, E_8\in M_8(SL_2(\mathbb{Z}))$.

Folsom: Proved mock theta conjectures using mock modularity but needed to compute a large number of coefficients

Andersen: Improved Folsom's proof so that one only needed to compute a small number of coefficients.

Topics List

- Ramanujan, First and Last Letters to Hardy, the Lost Notebook
- Partition function $p(n)$
- Dyson's rank of a partition
- Modular forms, theta functions, and mock theta functions
- Classical problems for mock theta functions
- *we are here*
- Coming Attractions
  - Appell–Lerch functions, Hecke-type double-sums, and a heuristic
  - Kronecker-type identities and sums of squares and triangular numbers
  - Threefield Identities, quantum modular forms
  - Modularity and the four-squares problem
  - Mock modularity and weak harmonic Maass forms