Моя Тетрадь

Lecture 2:

Appell-Lerch functions,
Hecke-type double-sums,
and a heuristic.

with applications to the identities for the 10th order mock theta functions found in Ramanujan's Lost Notebook

Lecture Plan

- motivation
  - mock theta functions have many forms
  - understanding the functions often involves changing from on form to another (ad hoc)
  - find a master formula
    take Hecke-type double-sums to Appell-Lerch functions

Lecture Plan

- Recall theta functions & Appell-Lerch fns.
  - state & prove a few basic properties
  - heuristic & new $m(x,q,z)$ properties
- Hecke-type double-sums
  - state & prove a few basic properties
  - heuristic & expanding Hecke-type double-sums in terms of $m(x,q,z)$ fn.

Lecture Plan

- Application

  use new $m(x,q,z)$ properties & Hecke-type double-sum expansions to give new proofs of Ramanujan's identities for the 10th order mock theta functions

* the four 10th order mock theta functions and their six identities were all found in Ramanujan's Lost Notebook.

Motivation - many forms of mock thetas

1) q-hypergeometric

$$f_0(q) := \sum_{n=0}^{\infty} q^{n^2} \Big/ (-q;q)_n$$

$$= 1 + \frac{q}{1+q} + \frac{q^4}{(1+q)(1+q^2)} + \cdots$$

2) Hecke-type double-sum

$$f_0(q) \prod_{n=1}^{\infty} (1-q^n)$$

$$= \sum_{n=0}^{\infty} q^{5n^2/2 + n/2} (1-q^{4n+2}) \sum_{j=-n}^{n} (-1)^j q^{-j^2}$$

Motivation - many forms of mock thetas

3) Appell-Lerch functions

$$f_0(q) = m(q^{14}, q^{30}, q^{14}) + m(q^{14}, q^{30}, q^{29}) + q^{-2} m(q^{4}, q^{30}, q^{4}) + q^{-2} m(q^{4}, q^{30}, q^{14})$$

where

$$m(x, q, z) := \frac{1}{j(z;q)} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{\binom{n}{2}} z^n}{1 - q^{n-1} x z}$$

$$j(z;q) := \sum_{n=-\infty}^{\infty} (-1)^n q^{\binom{n}{2}} z^n$$

4) Fourier Coefficients of meromorphic Jacobi forms (skip)

Motivation - changing forms

1) For the 5^{th} & 7^{th} order mock theta functions, Andrews used Bailey's Lemma to go from q-hypergeometric to Hecke-type double-sums

2) Andrews expressed the 5^{th} & 7^{th} order mock theta functions as Fourier coefficients of meromorphic Jacobi forms

Motivation - changing forms

3) Watson used a ${}_8\phi_7$ basic hypergeometric formula to convert from q-hypergeometric to $m(x,q,z)$ but this method only works for third orders.

- Recall some notation
- State ${}_8\phi_7$

Motivation - changing forms

$$(x)_n = (x;q)_n := \prod_{i=0}^{n-1} (1-q^i x)$$

$$(x_1, x_2, \ldots, x_k)_n := (x_1)_n (x_2)_n \cdots (x_k)_n$$

$$(x)_\infty = (x;q)_\infty = \prod_{i=0}^{\infty} (1-q^i x)$$

$$(x_1, x_2, \ldots, x_k)_\infty := (x_1)_\infty (x_2)_\infty \cdots (x_k)_\infty$$

Motivation – changing forms of ϕ_7

$$1+\sum_{n=1}^{\infty} \frac{(a, q\sqrt{a}, -q\sqrt{a}, b, c, d, e, f; q)_n}{(q, \sqrt{a}, -\sqrt{a}, aq/c, aq/e, aq/f, aq/g; q)_n} \cdot \left(a^2q^2/cdefg\right)^n$$

$$= \frac{(aq, aq/fg, aq/ge, aq/fe; q)_\infty}{(aq/e, aq/f, aq/g, aq/efg; q)_\infty} \cdot \left[1+\sum_{n=1}^{\infty} \frac{(aq/cd, e, f, g; q)_n}{(q, efg/a, aq/c, aq/d; q)_n} \cdot q^n\right]$$

$$f(q) := 1+\sum_{n=1}^{\infty} q^{n^2} / (-q;q)_n^2$$

$$f(q) = 4m(-q, q^3, q) + \prod_{n=1}^{\infty} \frac{(1-q^{6n-3})^2(1-q^{3n})}{(1-q^n)}$$

Motivation - changing forms

4) Hickerson used the constant term method to convert from Hecke-type double-sum to $m(x,q,z)$ form. This was key to his proof of the mock theta conjectures

Motivation - changing forms

Recall

$$j(z;q) = \sum_{n=-\infty}^{\infty} (-1)^n z^n q^{\binom{n}{2}}$$

$$B(z;q) := \frac{z^2 j(q^2;q^6) j(-z;q) j(z;q^3)}{j(z;q^2)}$$

Expanded B two ways

1st way: coefficient of z^1 is

$f_0(q)$ in Hecke-type double-sum form

2nd way: coefficient of z^1 is

$f_0(q)$ in $m(x,q,z)$ form

goal

- find new $m(x,q,z)$ properties
- expand Hecke-type double-sums in terms of $m(x,q,z)$ functions

Application

Give new proofs of the six identities for the four 10th order mock theta functions, all of which having been found in Ramanujan's Lost Notebook

Application Preview

The four tenth order mock thetas.

$$\phi(q) := \sum_{n=0}^{\infty} \frac{q^{\binom{n+1}{2}}}{(q;q^2)_{n+1}} \qquad \psi(q) := \sum_{n=0}^{\infty} \frac{q^{\binom{n+2}{2}}}{(q;q^2)_{n+1}}$$

$$X(q) := \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2}}{(-q;q)_{2n}} \qquad \chi(q) := \sum_{n=0}^{\infty} \frac{(-1)^n q^{(n+1)^2}}{(-q;q)_{2n+1}}$$

One of the six identities reads

$$\phi(q) - q^{-1}\psi(-q^4) + q^{-2}\chi(q^8)$$

$$= \frac{j(-q;q^2)\, j(-q^2;-q^{10})}{j(q^2;q^8)}$$

Celebrated results of

	two ids	InvMath '99
Choi	two ids	AdvMath '00
	two ids	ProcLMS '07

Application Preview

q-hypergeometric LHS

↓ Bailey's Lemma

Hecke-type double-sum

↓ Hecke-type to $m(x,q,z)$ formula

$m(x,q,z)$ form

↓ new $m(x,q,z)$ properties

theta functions RHS

$$(x)_n = (x;q)_n = \prod_{i=0}^{n-1} (1-q^i x)$$

$$(x)_\infty = (x;q)_\infty = \prod_{i=0}^{\infty} (1-q^i x)$$

$$j(x;q) := \sum_{n=-\infty}^{\infty} (-1)^n q^{\binom{n}{2}} x^n$$

$$\text{JTPid} \quad = (x;q)_\infty (q/x;q)_\infty (q;q)_\infty$$

$$j(x_1, x_2, \ldots, x_n; q)$$

$$:= j(x_1;q) j(x_2;q) \cdots j(x_n;q)$$

$$J_{a,m} := j(q^a;q^m) \quad \bar{J}_{a,m} := j(-q^a;q^m)$$

$$J_m = J_{m,3m} = \prod_{i=1}^{\infty} (1-q^{mi})$$

theta function

$$j(z;q) := \sum_{n=-\infty}^{\infty} (-1)^n z^n q^{\binom{n}{2}}$$

Partial theta function

$$\sum_{n=0}^{\infty} (-1)^n x^n q^{\binom{n}{2}}$$

False theta function

$$\sum_{n=-\infty}^{\infty} sg(n) (-1)^n q^{\binom{n}{2}} x^n$$

$$sg(n) := \begin{cases} 1 & n \geq 0 \\ -1 & n < 0 \end{cases}$$

Basic prop. of thetas

a) $j(x;q) = j(q/x;q)$

b) $j(q^n x;q) = (-1)^n q^{-\binom{n}{2}} x^{-n} j(x;q)$

c) $j(x;q) = 0 \iff x = q^n, n \in \mathbb{Z}$

a & c follow from triple product identity

e.g. c)

$$(x;q)_\infty = \prod_{i=0}^{\infty} (1 - xq^i)$$

if $x = q^0, q^{-1}, q^{-2}, \ldots$ $\uparrow = 0$

b) is next

$$j(q^n x;q) = \sum_{k=-\infty}^{\infty} (-1)^k q^{\binom{k}{2}} (q^n x)^k$$

$$= \sum_{k=-\infty}^{\infty} (-1)^k q^{\binom{k}{2} + nk} x^k$$

note $\binom{A+B}{2} = \binom{A}{2} + AB + \binom{B}{2}$

$$= \sum_{k=-\infty}^{\infty} (-1)^k q^{\binom{k+n}{2} - \binom{n}{2}} x^k$$

substitute $k \to k-n$

$$= \sum_{k=-\infty}^{\infty} (-1)^{k+n} q^{\binom{k}{2} - \binom{n}{2}} x^{k-n}$$

$$= (-1)^n q^{-\binom{n}{2}} x^{-n} \sum_{k=-\infty}^{\infty} (-1)^k q^{\binom{k}{2}} x^k$$

$$= (-1)^n q^{-\binom{n}{2}} x^{-n} j(x;q)$$

def Appell-Lerch fn.

If x, q, and z are nonzero complex numbers with $|q|<1$ and neither z nor xz equal to an integral power of q then

$$m(x,q,z) := \frac{1}{j(z;q)} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{\binom{n}{2}} z^n}{1-q^{n-1}xz}$$

Proposition

1) $m(x,q,z) = m(x,q,qz)$

2) $m(x,q,z) = x^{-1} m(x^{-1},q,z^{-1})$

3) $m(qx,q,z) = 1 - xm(x,q,z)$

Cor (a) $m(q,q^2,-1) = 1/2$

(b) $m(-1,q^2,q) = 0$

<u>Thm</u> $m(x,q,z_1) - m(x,q,z_0)$

$$= \frac{z_0 \bar{J}_1^3 j(z_1/z_0;q) j(xz_0z_1;q)}{j(z_0;q) j(z_1;q) j(xz_0;q) j(xz_1;q)}$$

Cor $m(x,q,z) = m(x,q,x^{-1}z^{-1})$

$$m(x,q,z) = m(x,q,qz)$$

def

$$m(x,q,qz) = \frac{1}{j(qz;q)} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{\binom{n}{2}} (qz)^n}{1-q^n xz}$$

ell trans prop $j(qz;q) = -z^{-1} j(z;q)$

$$= \frac{1}{-z^{-1} \cdot j(z;q)} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{\binom{n+1}{2}} z^n}{1-q^n xz}$$

$$= \frac{1}{j(z;q)} \sum_{n=-\infty}^{\infty} \frac{(-1)^{n+1} q^{\binom{n+1}{2}} z^{n+1}}{1-q^n xz}$$

subs $n \to n-1$

$$= \frac{1}{j(z;q)} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{\binom{n}{2}} z^n}{1-q^{n-1} xz}$$

$$= m(x,q,z)$$

want to show $m(x,q,z) = x^{-1} m(x^{-1},q,z)$

$$m(x,q,z) = \frac{1}{j(z;q)} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{\binom{n}{2}} z^n}{1 - q^{n-1} x z}$$

subs $n \to -n$

$$= \frac{1}{j(z;q)} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{\binom{n+1}{2}} z^{-n}}{1 - q^{-n-1} x z}$$

Recall

$$j(z;q) = j(q/z;q) = -z\, j(1/z;q)$$

$$m(x,q,z) = \frac{1}{j(z^{-1};q)} \sum_{n=-\infty}^{\infty} \frac{(-1)^{n+1} q^{\binom{n+1}{2}} z^{-n-1}}{1 - q^{-n-1} x z}$$

subs $n \to n-1$

$$= \frac{1}{j(z^{-1};q)} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{\binom{n}{2}} z^{-n}}{1 - q^{-n} x z}$$

$\Rightarrow$

multiply by $q^n x^{-1} z^{-1} / q^n x^{-1} z^{-1}$

$$= -\frac{1}{j(z^{-1};q)} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{\binom{n}{2}} q^n x^{-1} z^{-1}}{1 - q^n x^{-1} z^{-1}}$$

subs $n \to n-1$

$$= \frac{x^{-1}}{j(z^{-1};q)} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{\binom{n}{2}} z^{-n}}{1 - q^{n-1} x^{-1} z^{-1}}$$

$$= m(x^{-1}, q, z^{-1})$$

so $m(x, q, z) = x^{-1} m(x^{-1}, q, z^{-1})$

$$m(qx,q,z) = 1 - x\,m(x,q,z)$$

$$m(qx,q,z) = \frac{1}{j(z;q)} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{\binom{n}{2}} z^n}{1 - q^n x z}$$

trick

$$= \frac{1}{j(z;q)} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{\binom{n}{2}} z^n \left(1 - q^n x z + q^n x z\right)}{1 - q^n x z}$$

$$= \frac{1}{j(z;q)} \left(j(z;q) + x \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{\binom{n+1}{2}} z^{n+1}}{1 - q^n x z} \right)$$

subs $n \to n-1$

$$= 1 - \frac{x}{j(z;q)} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{\binom{n}{2}} z^n}{1 - q^{n-1} x z}$$

$$= 1 - x\,m(x,q,z)$$

Prop 1) $m(x,q,z) = m(x,q,qz)$

2) $m(x,q,z) = x^{-1} m(x^{-1},q,z^{-1})$

3) $m(qx,q,z) = 1 - x\, m(x,q,z)$

Cor a) $m(q,q^2,-1) = 1/2$

b) $m(-1,q^2,q) = 0$

Proof

a) $m(q,q^2,-1) = m(q^2 q^{-1},q^2,-1)$

(3) $= 1 - q^{-1} m(q^{-1},q^2,-1)$

(2) $= 1 - m(q,q^2,-1)$

$m(q,q^2,-1) = 1 - m(q,q^2,-1)$

$2m(q,q^2,-1) = 1$

$m(q,q^2,-1) = 1/2$

Proof (b)

$$m(-1, q^2, q) = -m(-1, q^2, q^{-1}) \quad (2)$$

$$= -m(-1, q^2, q) \quad (1)$$

compare far left with far right to have

$$m(-1, q^2, q) = -m(-1, q^2, q)$$

So $m(-1, q^2, q) = 0$

$F(x)$

$$= m(x,q,z_1) + m(x,q,z_0) - \frac{z_0 J_1^3\, j(z_1/z_0;q)\, j(xz_0z_1;q)}{j(z_0;q)\, j(z_1;q)\, j(xz_0;q)\, j(xz_1;q)}$$

Let q, z_0, z_1 be fixed

we wish to show $F(x)$ is identically zero

__claim__ $F(qx) = -xF(x)$

$$m(qx,q,z) = 1 - x\, m(x,q,z)$$

$$j(qx;q) = -x^{-1} j(x;q)$$

__claim:__ $F(x)$ is analytic for all $x \neq 0$

- at most simple poles at $x = q^n z_0^{-1}$ and $x = q^n z_1^{-1}$, $n \in \mathbb{Z}$
- residues at poles sum to zero

where do simple poles come from?

$$m(x,q_\lambda,z_1) = \frac{1}{j'(z_1;q_\lambda)} \sum_{k=-\infty}^{\infty} \frac{(-1)^k q_\lambda^{\binom{k}{2}} z_1^k}{1-q_\lambda^{k-1} x z_1}$$

poles happen when $x z_1 = q_\lambda^n$, $n \in \mathbb{Z}$

or equivalently $x = q_\lambda^n z_1^{-1}$ $n \in \mathbb{Z}$

similarly poles happen in $m(x,q_\lambda,z_0)$

when

$$x = q_\lambda^n z_0^{-1} \quad n \in \mathbb{Z}$$

poles occur in denominator of theta quotient i.e. when

$j(xz_0;q_\lambda) = 0$ or $j(xz_1;q_\lambda) = 0$

but both are equiv to

$$x = q_\lambda^n z_0^{-1} \quad n \in \mathbb{Z} \quad \text{or} \quad x = q_\lambda^n z_1^{-1} \quad n \in \mathbb{Z}$$

at the possible poles

the residues always sum to zero

So $F(x)$ is analytic for all $x \neq 0$

$F(x)$ can be written as a Laurent series in x valid for all $x \neq 0$

$$F(x) = \sum_{m \in \mathbb{Z}} c_m x^m$$

$$F(qx) = -x F(x)$$

$$\sum_{m \in \mathbb{Z}} c_m (qx)^m = -x \sum_{m \in \mathbb{Z}} c_m x^m$$

$$\Rightarrow c_{m+1} = -q^{-(1+m)} c_m$$

$$\Rightarrow c_m = (-1)^m q^{-\binom{m+1}{2}} c_0$$

So $F(x) = C_0 \sum_{m \in \mathbb{Z}} (-1)^m q^{-\binom{m+1}{2}} x^m$

if we assume $C_0 \neq 0$ the ratio test tells us $F(x)$ diverges for all $x \neq 0$ which is a contradiction! So $C_0 = 0$. and $F(x)$ is identically zero.

$$\lim_{m \to \infty} \left| \frac{X_{m+1}}{X_m} \right| = \lim_{m \to \infty} \left| \frac{q^{\binom{m+1}{2}} x}{q^{\binom{m+2}{2}}} \right|$$

$$= \lim_{m \to \infty} \left| \frac{x}{q^{m+1}} \right| \to \infty$$

the heuristic

$$m(q_\lambda x, q_\lambda, z) = 1 - x\, m(x, q_\lambda, z)$$

$$m(x, q_\lambda, z) = 1 - q_\lambda^{-1} x\, m(q_\lambda^{-1} x, q_\lambda, z)$$

iterate

$$m(x, q_\lambda, z) = 1 - q_\lambda^{-1} x\, m(q_\lambda^{-1} x, q_\lambda, z)$$

$$= 1 - q_\lambda^{-1} x + q_\lambda^{-3} x^2\, m(q_\lambda^{-2} x, q_\lambda, z)$$

$$= 1 - q_\lambda^{-1} x + q_\lambda^{-3} x^2 - q_\lambda^{-6} x^3\, m(\sim)$$

$$\sim 1 - q_\lambda^{-1} x + q_\lambda^{-3} x^2 - q_\lambda^{-6} x^3 + q_\lambda^{-10} x^4 + \sim$$

$$m(x, q_\lambda, z) \sim \sum_{r \geq 0} (-1)^r x^r q_\lambda^{-\binom{r+1}{2}}$$

*

Of course we cannot use an equal sign here because the infinite series on the right diverges for $|q|<1$.

However, it is useful to think of $m(x,q,z)$ as a partial or half theta function with q replaced by q^{-1}

let us break (*) into two parts depending on parity of r

$$m(x,q_\lambda,z) \sim \sum_{r\geq 0} (-1)^r q_\lambda^{-\binom{r+1}{2}} x^r \quad *$$

$$\sim \sum_{r\geq 0} q_\lambda^{-\binom{2r+1}{2}} x^{2r} - \sum_{r\geq 0} q_\lambda^{-\binom{2r+2}{2}} x^{2r+1}$$

$$\sim \sum_{r=0}^{\infty} (-1)^r q_\lambda^{-4\binom{r+1}{2}} (-q_\lambda x^2)^r$$

$$-q_\lambda^{-1} x \sum_{r=0}^{\infty} (-1)^r q_\lambda^{-4\binom{r+1}{2}} (-q_\lambda^{-1} x^2)^r$$

$$\sim m(-q_\lambda x^2, q_\lambda^4, z_0) - q_\lambda^{-1} x\, m(-q_\lambda^{-1} x^3, q_\lambda^4, z_1)$$

$$m(x,q,z) = m(-qx^2, q^4, z')$$

$$- q^{-1} x\, m(-q^{-1} x^2, q^4, z')$$

$$+ \frac{z' \overline{J}_2^3}{j(xz;q)\, j(-qx^2 z; q^2)} \left[\frac{j(-qx^2 z z'; q^2)\, j(z^2/z'; q^4)}{j(z'; q^4)\, j(z, q^2)} \right.$$

$$\left. - xz \frac{j(-q^2 x^2 z z'; q^2)\, j(q^2 z^2/z'; q^4)}{j(z'; q^4)\, j(qz; q^2)} \right]$$

later we will use

$$D_2(x,q,z,z')$$

$$:= m(x,q,z) - m(-qx^2, q^4, z')$$

$$+ q^{-1} x\, m(-q^{-1} x^2, q^4, z')$$

The heuristic $m(x,q,z) \sim \sum_{r=0}^{\infty} (-1)^r x^r q^{-\binom{r+1}{2}}$

This suggest several other functional equations that $m(x,q,z)$ might satisfy.
Most of these eqns are not true but become true when an appropriate theta function is added.

Roughly speaking we may think of '$\sim$' as 'mod theta'

If we break the sum into n parts depending on the value of r mod n we find

$$m(x,q,z) \sim \sum_{r=0}^{n-1} (-1)^r q^{-\binom{r+1}{2}} x^r \cdot m\left((-1)^{n+1} q^{\binom{n}{2}-nr} x^n, q^{n^2}, z_r\right)$$

So we expect the difference between the two sides to be a theta fn which does turn out to be the case.

$$m(x,q,z)$$

$$= \sum_{r=0}^{n-1} q^{-\binom{r+1}{2}} (-x)^r \cdot m\left(-q^{\binom{n}{2}-nr}(-x)^n, q^{n^2}, z'\right)$$

$$+ \frac{z' J_n^3}{j(xz;q)\, j(z';q^{n^2})} \cdot \sum_{r=0}^{n-1} \left[q^{\binom{r}{2}} (-xz)^r \cdot j\left(-q^{\binom{n}{2}+r}(-x)^n z z'; q^n\right) \cdot \frac{j\left(q^{nr} z^n / z'; q^{n^2}\right)}{j\left(-q^{\binom{n}{2}}(-x)^n z'; q^n\right) j\left(q^r z; q^n\right)} \right]$$

Hecke-type double-sums

$$\sum_{sg(r)=sg(s)} sg(r)\, c_{r,s} \qquad sg(r) = \begin{cases} 1 & \text{if } r \geq 0 \\ -1 & \text{if } r < 0 \end{cases}$$

other was to write this

$$\left(\sum_{r,s \geq 0} - \sum_{r,s<0} \right) c_{r,S}$$

Properties

1) $\displaystyle\sum_{sg(r)=sg(s)} sg(r)\, c_{r,s} = -\sum_{sg(r)=sg(s)} sg(r)\, c_{-1-r,-1-s}$

2) if R, S are integers

$$\sum_{sg(r)=sg(s)} sg(r) c_{r,s}$$

$$= \sum_{sg(r)=sg(S)} sg(r) c_{r+R, s+S}$$

$$+ \sum_{r=0}^{R-1} \sum_{s\in\mathbb{Z}} c_{r,s} + \sum_{s=0}^{S-1} \sum_{r\in\mathbb{Z}} c_{r,s}$$

sum convention for $b<a$

$$\sum_{r=a}^{b} c_r = -\sum_{r=b+1}^{a-1} c_r$$

$a=0$, $b=1$

$$\sum_{r=0}^{-1} c_r = -\sum_{r=0}^{-1} c_r = 0$$

$$C_{r,s} = (-1)^{r+s} x^r y^s q_\lambda^{a\binom{r}{2} + brs + c\binom{s}{2}}$$

$$\sum_{sg(r)=sg(s)} sg(r) C_{r,s}$$ becomes

$f_{a,b,c}(x,y,q_\lambda)$

$$:= \left(\sum_{r,s \geq 0} - \sum_{r,s<0} \right) (-1)^{r+s} x^r y^s \cdot q_\lambda^{a\binom{r}{2} + brs + c\binom{s}{2}}$$

$a,b,c > 0$ converges abs

All of Ramanujan's classical mock theta functions can be written in terms of $f_{a,b,c}(x,y,q_\lambda)$'s.

1) $f_{a,b,c}(x,y,q)$

$$= -\frac{q^{a+b+c}}{xy} f_{a,b,c}\left(q^{2a+b}x,\, q^{2c+b}y,\, q\right)$$

2) $f_{a,b,c}(x,y,q)$

$$= (-1)^{\ell+k} x^{\ell} y^{k} q^{a\binom{\ell}{2}+b\ell k+c\binom{k}{2}}$$

$$\cdot f_{a,b,c}\left(q^{a\ell+bk}x,\, q^{b\ell+ck}y,\, q\right)$$

$$+ \sum_{m=0}^{\ell-1} (-1)^m x^m q^{a\binom{m}{2}} j\left(q^{mb}y; q^{c}\right)$$

$$+ \sum_{m=0}^{k-1} (-1)^m y^m q^{c\binom{m}{2}} j\left(q^{mb}x; q^{a}\right)$$

how do we use heuristic to understand $f_{a,b,c}$???

$$m(x,q,z) \sim \sum_{r \geq 0} (-1)^r q^{-\binom{r+1}{2}} x^r \qquad (*)$$

If we iterate a functional equation for some function f and we see divergent partial theta series like the r.h.s of $(*)$ then we may be able to write $f(x)$ in terms of the $m(x,q,z)$ function.

In our functional eqn for $f_{a,b,c}$ we let $l, k \to \infty$

We simplify our example by letting $a=1, b=2, c=1$.

$f_{1,2,1}(x,y,q)$

$$\sim \sum_{r=0}^{\infty} (-1)^r x^r q^{\binom{r}{2}} j(yq^{2r};q)$$

$$+ \sum_{s=0}^{\infty} (-1)^s y^s q^{\binom{s}{2}} j(xq^{2s};q)$$

$$j(q^n x;q) = (-1)^n q^{-\binom{n}{2}} x^{-n} j(x;q)$$

we unwind $j(yq^{2r};q)$ by using

elliptic transformation property

$$j(q^n x;q) = (-1)^n q^{-\binom{n}{2}} x^{-n} j(x;q)$$

$$\sum_{m=0}^{\infty} (-1)^m x^m q^{\binom{m}{2}} j(q^{m2} y; q)$$

$$\sim j(y;q) \sum_{m=0}^{\infty} (-1)^m x^m q^{\binom{m}{2}} (-1)^{2m} \cdot q^{-\binom{2m}{2}} y^{-2m}$$

$$j(y;q) \sum_{m=0}^{\infty} (-1)^m \left(\frac{q^2 x}{y^2} \right)^m q^{-3\binom{m+1}{2}}$$

$$\sim j(y;q)\, m\left(q^2 x / y,\, q^3,\, *\right)$$

$$\mathcal{L}_{1,2,1}(x,y,q)$$

$$= j(y;q)\, m(q^2 x/y^2, q^3, -1)$$

$$+ j(x;q)\, m(q^2 y/x^2, q^3, -1)$$

$$- y \frac{1}{j(-1;q^3)} \frac{\overline{J}_3^3\, j(-x/y;q)\, j(q^2 xy;q^3)}{j(-qy^2/x;q^3)\, j(-qx^2/y;q^3)}$$

Examples

$$f_{1,2,1}(q,q,q)$$

$$= j(q;q)\, m(q, q^3, -1)$$

$$+ j(q;q)\, m(q, q^3, -1) + J_1^2$$

$j(q;q) = 0$

so $$f_{1,2,1}(q,q,q) = J_1^2$$

$$= \prod_{i=1}^{\infty} (1-q^i)^2$$

Examples

6th order mock theta fn

$$\overline{J}_{1,4}\,\phi(q) = f_{1,2,1}(q,-q,q)$$

Bailey Pair Technology

$$f_{1,2,1}(q,-q,q)$$

$$= j(-q;q)\,m(q,q^3,-1)$$

$$+ j(q;q)\,m(-q,q^3,-1) + 0$$

$$= j(-q;q)\,m(q,q^3,-1)$$

or $\phi(q) = 2m(q,q^3,-1)$

МОЯ ТЕТРАД

identitces for the
four 10th order mock
theta functions

sketch of proofs

Choi proved all six id's for Ramanujan's 10th order mock theta functions

$$\phi(q) := \sum_{n=0}^{\infty} \frac{q^{\binom{n+1}{2}}}{(q;q^2)_{n+1}} \qquad \psi(q) := \sum_{n=0}^{\infty} \frac{q^{\binom{n+2}{2}}}{(q;q^2)_{n+1}}$$

$$X(q) := \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2}}{(-q;q)_{2n}} \qquad \chi(q) := \sum_{n=0}^{\infty} \frac{(-1)^n q^{(n+1)^2}}{(-q;q)_{2n+1}}$$

Two of the six ids read

$$\phi(q) - q^{-1}\psi(-q^4) + q^{-2}\chi(q^8) = \frac{j(-q;q^2)\, j(-q^2;-q^{10})}{j(q^2;q^8)}$$

$$\psi(q) + q\phi(-q^4) + X(q^8) = \frac{j(-q;q^2)\, j(-q^4;-q^{20})}{j(q^2;q^8)}$$

Notation

$$\bar{J}_{a,m} := j(-q_\lambda^{a}; q_\lambda^{m})$$

$$J_{a,m} := j(q_\lambda^{a}; q_\lambda^{m})$$

$$\bar{J}_m = \bar{J}_{m,3m} = j(q_\lambda^{m}; q_\lambda^{3m})$$

$$f_{a,b,c}(x,y,z)$$

$$:= \left(\sum_{r,s\geq 0} - \sum_{r,s<0}\right)(-1)^{r+s} x^r y^s q_\lambda^{a\binom{r}{2} + brs + c\binom{s}{2}}$$

note weight system depends on which region we sum over!

Using Bailey's Lemma Choi showed

$J_{1,2}\,\phi(q) = f_{2,3,2}(q^2, q^2, q)$

$J_{1,2}\,\psi(q) = -q^2 f_{2,3,2}(q^4, q^4, q)$

$\overline{J}_{1,4}\,X(q) = f_{2,3,2}(-q^3, -q^3, q^2)$

$\overline{J}_{1,4}\,(2 - X(q)) = q\, f_{2,3,2}(-q^{-1}, -q^{-1}, q^2)$

We then use a formula to convert from $f_{2,3,2}(x, y, q)$

to $m(x, q, z)$'s.

$$f_{2,3,2}(x,y,q) := \left(\sum_{r,s\geq 0} - \sum_{r,s<0} \right) (-1)^{r+s} x^r y^s q^{2\binom{r}{2}+3rs+2\binom{s}{2}}$$

$$\begin{aligned} f_{2,3,2}(x,y,q) &= j(x;q^2)\, m(q^6y^2/x^3, q^{10}, -1) \\ &\quad - y\, j(q^3x;q^2)\, m(qy^2/x^3, q^{10}, -1) \\ &\quad + j(y;q)\, m(q^6x^2/y^3, q^{10}, -1) \\ &\quad - x\, j(q^3y;q^2)\, m(qx^2/y^3, q^{10}, -1) \\ &\quad - \frac{1}{J_{0,10}} \cdot \frac{y}{qx} \cdot \frac{J_5^3\, j(-x^2/y^2;q^2)\, j(q^3xy;q^5)}{j(-q^4y^3/x^2;q^5)\, j(-q^4x^3/y^2;q^5)} \end{aligned}$$

Take $f_{2,3,2}(x,y,q) \to m(x,q,z)$

$$\phi(q) = -q^{-1} m(q, q^{10}, q) - q^{-1} m(q, q^{10}, q^{2})$$

$$\psi(q) = -m(q^{3}, q^{10}, q) - m(q^{3}, q^{10}, q^{3})$$

$$\mathcal{X}(q) = m(-q^{2}, q^{5}, q) + m(-q^{2}, q^{5}, q^{4})$$

$$\chi(q) = m(-q, q^{5}, q^{2}) + m(-q, q^{5}, q^{3})$$

Three-term Weierstrass Relation

$$j(ac, a/c, bd, b/d; q)$$

$$= j(ad, a/d, bc, b/c; q)$$

$$+ b/c \cdot j(ab, a/b, cd, c/d; q)$$

$$j(x_1, x_2, \ldots, x_n; q)$$
$$= j(x_1; q)\, j(x_2; q) \cdots j(x_n; q)$$

$$m(x,q,z) := \frac{1}{j(z;q)} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{\binom{n}{2}} z^n}{1 - q^{n-1} xz}$$

$$j(z;q) := \sum_{n=-\infty}^{\infty} (-1)^n q^{\binom{n}{2}} z^n$$

$$m(x,q,z) = m(-qx^2, q^4, z') - q^{-1} x\, m(-q^{-1}x^2, q^4, z')$$
$$+ \frac{z' J_2^3}{j(xz;q)\, j(-qx^2 z'; q^2)} \cdot \left[\frac{j(-qx^2 z z'; q^2)\, j(q^2/z'; q^4)}{j(z'; q^4)\, j(z; q^2)} - xz \frac{j(-q^2 x^2 z z'; q^2)\, j(q^2 z^2/z'; q^4)}{j(z'; q^4)\, j(qz; q^2)} \right]$$

$$D_2(x,q,z,z') := m(x,q,z) - m(-qx^2, q^4, z') + q^{-1} x\, m(-q^{-1}x^2, q^4, z')$$

$D_2(x, q, z, z')$ is a sum of quotients of theta functions

LHS

$$\psi(q) + q\phi(-q^4) + \overline{X}(q^8)$$

= expression $f_{2,3,2}(x, y, q)$'s

= expression in $m(x, q, z)$'s

$$= -D_2(q^3, q^{10}, q^6, q^{-8}) - D_2(q^3, q^{10}, q^4, q^8)$$

$$= \frac{-\overline{J}_{20}^3 \overline{J}_{14,30} \overline{J}_{20,40}}{J_{1,10} \overline{J}_{8,40} \overline{J}_{8,20} J_{6,20}} - q \cdot \frac{\overline{J}_{20}^3 \overline{J}_{18,20} \overline{J}_{20,40}}{J_{7,10} J_{8,40} \overline{J}_{4,20} J_{6,20}}$$

$$= j(-q; q^2)\, j(-q; -q^{10}) \Big/ j(q^2; q^8)$$ RHS

*three-term Weierstrass Reln

$$D_3(x, q_\lambda, z, z')$$

$$:= m(x, q_\lambda, z)$$

$$- m(q_\lambda^3 x^3, q_\lambda^q, z')$$

$$+ q_\lambda^{-1} x \, m(x^3, q_\lambda^q, z')$$

$$- q_\lambda^{-3} x^2 \, m(q_\lambda^{-3} x^3, q_\lambda^q, z')$$

$$= \sum_{r=0}^{2} \mathrm{theta}(r, n, x, q_\lambda, z, z')$$

(m-split $n = 3$)